Convexity Theorem for Hyperbolic Functions in Complex Analysis

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Abstract. This paper studies the convexity of hyperbolic complex functions, where hyperbolic numbers are commutative rings that contain zero divisors and are composed of two real numbers. Based on the zero-divisor factorization theorem of hyperbolic numbers and other properties of functions of hyperbolic numbers, this article establishes necessary and sufficient conditions for the convexity of $Cl_{0,1}$ -differentiable functions with hyperbolic complex variables. This study, which generalizes the convexity characterization theorems from real analysis to the hyperbolic complex plane, will further establish a theoretical research foundation for the function theory of hyperbolic complex analysis and meanwhile provide impetus for the application of hyperbolic functions in physics.

Keywords: Hyperbolic Numbers, Convex Functions, Partial Order.

1. Introduction

Luna defined the hyperbolic numbers D based on the hyperbolic imaginary unit k and pointed out that it is a commutative ring [1]. After the above definition, Abouricha, Lino and other authors defined orthogonal idempotent elements and provided the relevant properties of orthogonal idempotent elements [2-3]. On these bases, Matsui and Sato provided definitions for non-negative cone and non-positive cone. Afterwards, they introduced a partial order, symbol, and thus provided the definition of intervals on hyperbolic numbers. After the definition of orthogonal idempotent elements was established, they provided the definition of hyperbolic number modulus and related properties based on it [4].

Lino with other authors derived the definition of the function of double and complex numbers based on the definite guidance on the idempotent plane, which lays the foundation for the subsequent research [5]. Kuloglu, Ozkan and Engin proposed and studied the extension of the classical hyperbolic functions. In addition, many of the characteristics of the k -Pell hyperbolic function is given. Finally, some of the graphs and surfaces related to the k -Pell hyperbolic function is introduced [6]. Mirosaw and Kazimierz proved theorems contrary to Ohlin's lemma for convex and strongly convex functions. In addition to this, they also provided new proofs of the probabilistic characteristics of convex and strongly convex functions [7]. Crouzeix, Hassouni, Giannessi and other authors obtained the second-order differentiability of the marginal function of the convex two-differentiable function, and combined the convexity and differentiability in this way [8].

Ankita, Rupakshi and Kottak Karan introduced a new double complex generalization of the Hurwitz - Lerch zeta function. They used a new generalized form of the beta function involving the Appell series and the Lauricella function. They also developed a new form of generalized fractional kinetic equations and obtained its solution using natural transformations [9]. Ortega, Yesenia, Ocampo with other authors proposed double plural versions of Cousin's problems, and their relationship to the double plural versions of Weierstrass and Mittag-Leffler's theorem. After that, they established a relationship between these theorems and the cousin problem [10]. Ghazala, Masoumeh, Saima and Muhammad extended the hyperbolic function method to construct traveling wave solutions. By using the proposed method in the paraxial equation, different types of soliton solutions can be recovered, such as dark solitons, bright solitons, and periodic solutions [11]. Hussain, Ibrahim, Birkea and other authors proposed a new extended hyperbolic function and provided its hyperbolic solutions. It is found that the model exhibits periodic oscillation nonlinear waves, kink wave profiles, multiple soliton profiles, singular solutions, mixed singular solutions, and mixed hyperbolic solutions [12].

2. Preliminaries

Hyperbolic numbers are commutative rings containing zero divisors. Their algebraic structure is similar to that of complex numbers, and they are defined as follows:

$$D := \{\varepsilon = x + ky : x, y \in \mathbb{R}\}. \tag{1}$$

Among them, k is the unit of hyperbolic numbers and satisfies $k^2 = 1$ and $k \neq \pm 1$, These numbers have alternative names like double, spacetime, perplex, or split complex numbers. This ring is commutative under addition and multiplication:

$$\varepsilon_1 + \varepsilon_2 = (x_1 + x_2) + k(y_1 + y_2), \tag{2}$$

$$\varepsilon_1 \varepsilon_2 = (x_1 x_2 + y_1 y_2) + k(x_1 y_2 + x_2 y_1). \tag{3}$$

For $\varepsilon = x + ky$, the real part is $\operatorname{Re}(\varepsilon) = x$, the hyperbolic part is $\operatorname{Im}(s) = y$, and the conjugate is $\overline{\varepsilon} = x - ky$.

Hyperbolic numbers admit a decomposition into orthogonal idempotent elements:

$$e = \frac{1+k}{2},\ e^{\dagger} = \frac{1-k}{2}. \tag{4}$$

Satisfying $e^2 = e$, $\left(e^{\dagger}\right)^2 = e^{\dagger}$, $ee^{\dagger} = 0$, $e + e^{\dagger} = 1$ and $e - e^{\dagger} = k$, For any element ε belonging to the set D, it can be expressed in the form:

$$\varepsilon = (x + y)e + (x - y)e^{\dagger} = \alpha e + \beta e^{\dagger}. \tag{5}$$

When we perform a standard basis decomposition of hyperbolic numbers into e and $e^{\dagger}$ dimensions, a hyperbolic number exhibits the following decomposition property:

$$D = \{\alpha e + \beta e^{\dagger} \mid \alpha, \beta \in \mathbb{R}\}. \tag{6}$$

Figure 1 effectively illustrates that the hyperbolic number ζ is a linear combination of e and $e^{\dagger}$

Zero divisors in D are scalar multiples of e or $e^{\dagger}$, lying on the null lines $y = \pm x$. Therefore, ε is a zero divisor if and only if $\varepsilon = xe$ or $\varepsilon = xe^{\dagger}$, where $x, y \in \mathbb{R} - \{0\}$. These elements are called the light cone and we denote it by L. Additionally, for $\varepsilon_1 = \alpha_1 e + \alpha_2 e^{\dagger}$ and $\varepsilon_2 = \beta_1 e + \beta_2 e^{\dagger}$ which belong to D, we have:

$$\varepsilon_1 + \varepsilon_2 = (\alpha_1 + \beta_1)e + (\alpha_2 + \beta_2)e^{\dagger}, \tag{7}$$

$$\varepsilon_1 \varepsilon_2 = (\alpha_1 \beta_1)e + (\alpha_2 \beta_2)e^{\dagger}. \tag{8}$$

The partial ordering of the set D is defined as follows, and it is illustrated in Figure 1 below:

Non-negative cone:

$$D^{+}=\left\{\alpha e+\beta e^{+} \mid \alpha \geq 0, \beta \geq 0\right\}. \tag{9}$$

Non-positive cone:

$$D^{-}=\left\{\alpha e+\beta e^{+} \mid \alpha \leq 0, \beta \leq 0\right\}. \tag{10}$$

A positive hyperbolic number is defined as follows:

$$D^{+}=\left\{x+ky \mid x \geq 0; |y| \leq x\right\}. \tag{11}$$

This means that x is greater than 0 and the magnitude of the second component y of the hyperbolic number is less than x. Similarly, we define the set of negative hyperbolic numbers using a similar approach:

$$D^{-}=\left\{x+ky \mid x \leq 0; |y| \leq |x|\right\}. \tag{12}$$

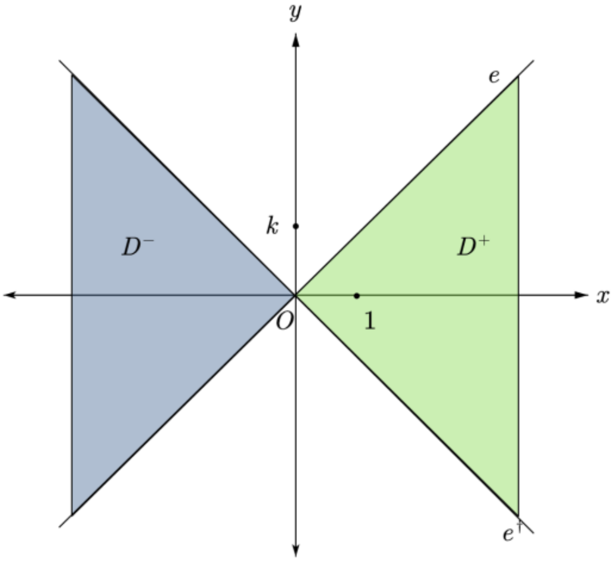

Figure 1. Positive and hyperbolic numbers and negative hyperbolic numbers

When the aforementioned conditions are met, if one of the components is 0, we refer to it as a semi-positive or semi-negative hyperbolic number. Meanwhile, for the remaining hyperbolic numbers that do not satisfy the aforementioned conditions, they cannot be called positive or negative hyperbolic numbers.

A partial order $\preceq$ is induced:

$$\varepsilon_1 \preceq \varepsilon_2 \Leftrightarrow \varepsilon_2-\varepsilon_1 \in D^{+}. \tag{13}$$

Equivalently, for $\varepsilon_1=\alpha_1 e+\beta_1 e^{\dagger}$ and $\varepsilon_2=\alpha_2 e+\beta_2 e^{\dagger}$:

$$\varepsilon_1 \preceq \varepsilon_2 \Leftrightarrow \alpha_1 \leq \alpha_2 \text{ and } \beta_1 \leq \beta_2. \tag{14}$$

This order is reflexive, transitive, and antisymmetric. A closed hyperbolic interval $\left[\varepsilon_1, \varepsilon_2\right]_D$ is defined as follows, and Figure 2 illustrates its definition:

$$\left[\varepsilon_1, \varepsilon_2\right]_D=\left\{u \in D \mid \varepsilon_1 \preceq u \preceq \varepsilon_2\right\}. \tag{15}$$

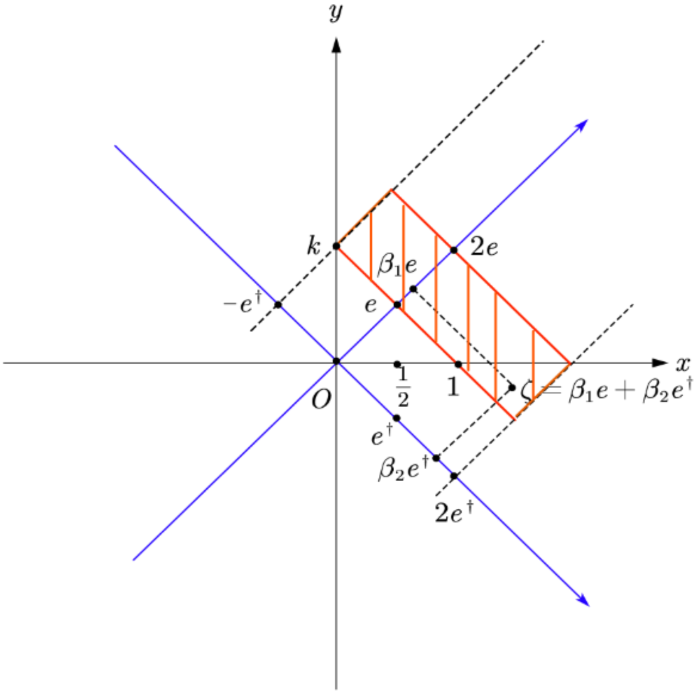

Figure 2. The hyperbolic interval $[k, 2]_D$

Degenerate interval occurs if $\varepsilon_2 - \varepsilon_1$ is a non-negative zero divisor and non-degenerate interval occurs if $\varepsilon_2 - \varepsilon_1$ is invertible. The length of a hyperbolic interval is a non-negative hyperbolic number.

For a hyperbolic number $\varepsilon = x + ky$, two distinct modulus definitions exist hyperbolic modulus (A positive hyperbolic number derived from the idempotent basis) and real-valued modulus (A positive real number aligned with Lorentzian geometry) and the second definition was preferred. Givens $\varepsilon = \alpha e + \beta e^\dagger$ in its idempotent basis representation, the hyperbolic modulus is:

$$|\varepsilon|_k = |\alpha|_k e + |\beta|_k e^\dagger. \tag{16}$$

Satisfying:

(i). Non-degeneracy:

$$|\varepsilon|_k = 0 \Leftrightarrow \varepsilon = 0. \tag{17}$$

(ii). Multiplicativity:

$$|\varepsilon_1 \varepsilon_2|_k = |\varepsilon_1|_k |\varepsilon_2|_k. \tag{18}$$

(iii). Subadditivity:

$$|\varepsilon_1 + \varepsilon_2|_k \preceq |\varepsilon_1|_k + |\varepsilon_2|_k. \tag{19}$$

3. Analysis of Functions of Hyperbolic numbers

We examine functions defined on an open domain $U \subseteq D$:

$f : U \to D.$

Similar to complex functions, functions defined on the hyperbolic complex plane also map the hyperbolic number plane to another hyperbolic number plane.

Expressed in terms of real and hyperbolic components as $f(z)=f_1(x,y)+kf_2(x,y)$, where $f_1,f_2\in C(U)$. Central to this analysis is the characterization of differentiability via the existence of the limit:

$$\lim_{\substack{h\to 0\\ h\notin L}}\frac{f(z_0+h)-f(z_0)}{h}, \tag{20}$$

Where L denoting a negligible subset excluded to avoid degenerate limits.

Case 1: Real Directional Derivative $(h=h_x)$

When the increment h is restricted to the real axis, the derivative simplifies to:

$$\lim_{h_x\to 0}\frac{f(z_0+h_x)-f(z_0)}{h_x}=\frac{\partial f_1}{\partial x}(x_0,y_0)+k\frac{\partial f_2}{\partial x}(x_0,y_0). \tag{21}$$

Case 2: Hyperbolic Directional Derivative $(h=kh_y)$

For a purely hyperbolic increment $h=kh_y$, the derivative becomes:

$$\lim_{kh_y\to 0}\frac{f(z_0+kh_y)-f(z_0)}{kh_y}=k\frac{\partial f_1}{\partial x}(x_0,y_0)+\frac{\partial f_2}{\partial x}(x_0,y_0). \tag{22}$$

Definition 3.1 A function f defined on a hyperbolic interval $[\varpi,\delta]_D$ is said to be $Cl_{0,1}$-differentiable and hyperbolically convex if, for all $\forall\varepsilon_1,\varepsilon_2\in[\varpi,\delta]_D$ and $\lambda\in(0,1)_D$, the following inequality holds:

$$f(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2)\preceq\lambda f(\varepsilon_1)+(1-\lambda)f(\varepsilon_2). \tag{23}$$

Lemma 3.1 A split-complex function f is $Cl_{0,1}$ -differentiable if and only if it decomposes into:

$$f=f_1(\alpha)e+f_2(\beta)e^{\dagger}. \tag{24}$$

Where e and $e^{\dagger}$ represent the idempotent basis elements of the split-complex algebra as illustrated in Figure 3. Conversely, f is $Cl_{0,1}$-ant differentiable if and only if:

$$f=f_1(\beta)e+f_2(\alpha)e^{\dagger}. \tag{25}$$

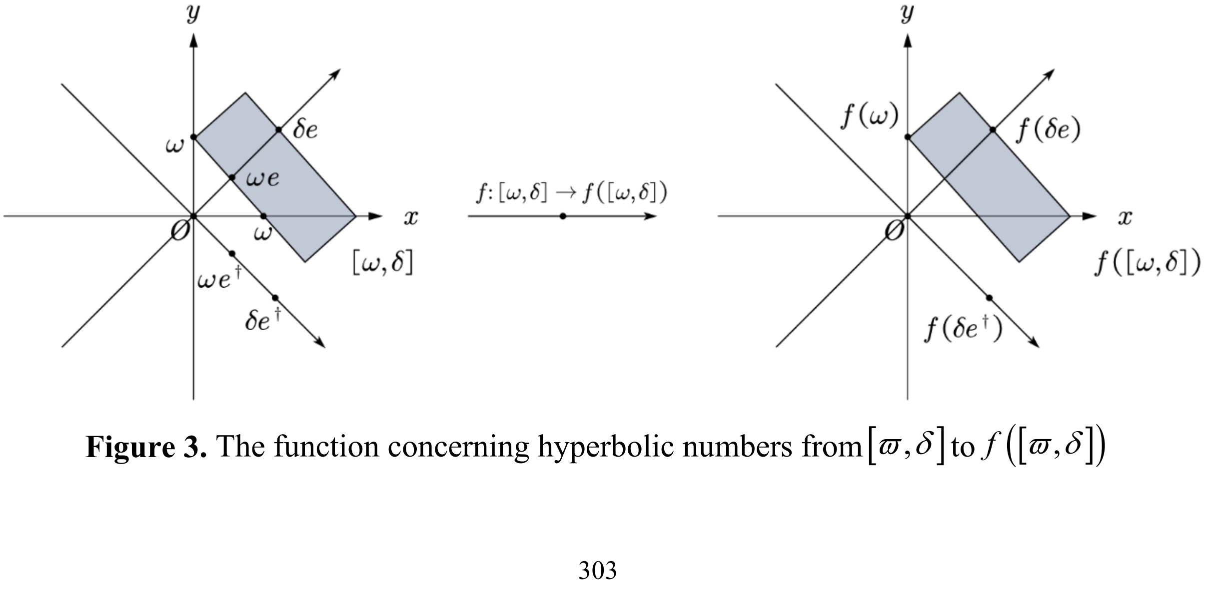

Figure 3. The function concerning hyperbolic numbers from $[\varpi,\delta]$ to $f([\varpi,\delta])$

This criterion generalizes the concept of holomorphicity to the split-complex plane D, enabling the identification of hyperbolic analogues of classical complex functions. By enforcing compatibility between partial derivatives and algebraic structure, the framework rigorously extends analytic tools to this geometric algebra context.

Theorem 3.1 Let f be a $Cl_{0,1}$-differentiable function on the hyperbolic interval $[\varpi,\delta]_D$, Consider the following elements: $\varepsilon_1=\alpha_1 e+\beta_1 e^{\dagger}\in[\varpi,\delta]_D$, $\varepsilon_2=\alpha_2 e+\beta_2 e^{\dagger}\in[\varpi,\delta]_D$, $\varepsilon=\alpha e+\beta e^{\dagger}\in[\varpi,\delta]_D$, $\lambda=\lambda_1 e+\lambda_2 e^{\dagger}\in(0,1)_D$ where $\alpha,\alpha_1,\alpha_2,\beta,\beta_1,\beta_2,\lambda_1,\lambda_2\in\mathbb{R}$. if f satisfies the convexity condition:

$$f\left(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2\right)\preceq\lambda f(\varepsilon_1)+(1-\lambda)f(\varepsilon_2), \tag{26}$$

Then this condition is equivalent to the non-negativity of the second derivative for all $\varepsilon\in[\varpi,\delta]_D$:

$$f''(\varepsilon)=f_1''(\alpha)e+f_2''(\beta)e^{\dagger}\succeq 0. \tag{27}$$

Proof: First, we prove the necessity: From property (4), we have:

$$\lambda\varepsilon_1+(1-\lambda)\varepsilon_2 \tag{28}$$

$$=\left(\lambda_1 e+\lambda_2 e^{\dagger}\right)\left(\alpha_1 e+\beta_1 e^{\dagger}\right)+\left(1-\lambda_1 e-\lambda_2 e^{\dagger}\right)\left(\alpha_2 e+\beta_2 e^{\dagger}\right). \tag{29}$$

Then since $e^2=e$, $\left(e^{\dagger}\right)^2=e^{\dagger}$, and $ee^{\dagger}=0$, the above expression simplifies to:

$$=\lambda_1\alpha_1 e+\lambda_2\beta_1 e^{\dagger}+\alpha_2 e-\lambda_1\alpha_2 e+\beta_2 e^{\dagger}-\lambda_2\beta_2 e^{\dagger} \tag{30}$$

$$=\left[\lambda_1\alpha_1+(1-\lambda_1)\alpha_2\right]e+\left[\lambda_2\beta_1+(1-\lambda_2)\beta_2\right]e^{\dagger}. \tag{31}$$

If f is $Cl_{0,1}$-differentiable, by Lemma 3.1, the left-hand side of the partial order inequality can be decomposed as:

$$f\left(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2\right) \tag{32}$$

$$=f\left(\left[\lambda_1\alpha_1+(1-\lambda_1)\alpha_2\right]e+\left[\lambda_2\beta_1+(1-\lambda_2)\beta_2\right]e^{\dagger}\right) \tag{33}$$

$$=f_1\left(\lambda_1\alpha_1+(1-\lambda_1)\alpha_2\right)e+f_2\left(\lambda_2\beta_1+(1-\lambda_2)\beta_2\right)e^{\dagger}. \tag{34}$$

Similarly, using the same properties, the right-hand side of the partial order inequality becomes:

$$\lambda f(\varepsilon_1)+(1-\lambda)f(\varepsilon_2) \tag{35}$$

$$=\left(\lambda_1 e+\lambda_2 e^{\dagger}\right)\left(f_1(\alpha_1)e+f_2(\beta_1)e^{\dagger}\right)+\left(1-\lambda_1 e-\lambda_2 e^{\dagger}\right)\left(f_1(\alpha_2)e+f_2(\beta_2)e^{\dagger}\right) \tag{36}$$

$$=\left[\lambda_1 f_1(\alpha_1)+(1-\lambda_1)f_1(\alpha_2)\right]e+\left[\lambda_2 f_2(\beta_1)+(1-\lambda_2)f_2(\beta_2)\right]e^{\dagger}. \tag{37}$$

By the inequality $f\left(\lambda\varepsilon_1+(1-\lambda)\varepsilon_2\right)\preceq\lambda f(\varepsilon_1)+(1-\lambda)f(\varepsilon_2)$ and equation (15), we have:

$$f_1\left(\lambda_1\alpha_1+(1-\lambda_1)\alpha_2\right)\preceq\lambda_1 f_1(\alpha_1)+(1-\lambda_1)f_1(\alpha_2) \tag{38}$$

$$f_2\left(\lambda_2\beta_1+(1-\lambda_2)\beta_2\right)\preceq\lambda_2 f_2(\beta_1)+(1-\lambda_2)f_2(\beta_2). \tag{39}$$

According to Definition 3.1 since f_1 and f_2 are convex functions, their second derivatives satisfy $f_1''(\alpha) \succeq 0$ and $f_2''(\beta) \succeq 0$, Therefore, we conclude $f''(\varepsilon) = f_1''(\alpha)e + f_2''(\beta)e^{\dagger} \succeq 0$, Similarly, the sufficiency can be proven.

4. Conclusion

This paper investigates the theory of convexity for functions on the hyperbolic plane and establishes necessary and sufficient conditions for determining the convexity of $Cl_{0,1}$-differentiable functions with hyperbolic complex variables. Since hyperbolic numbers form a commutative ring with zero divisors generated by two real numbers, they represent a more complex algebraic structure. By using the decomposition properties of hyperbolic functions, this paper derives a convexity theorem for hyperbolic functions in complex. However, this paper only studies the convexity of the functions of hyperbolic complex variables and the related conclusions, without involving the concavity of the functions of hyperbolic complex variables and the related results. So, in the next step, it is possible to study the concavity of the functions of hyperbolic complex variables and obtain results similar to those in this paper. This work further lays a theoretical foundation for the application of hyperbolic numbers in physics, injecting significant momentum into the study of four-dimensional spacetime physical contexts.

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