Research on the Movement Patterns and Optimal Turning Paths of Bench Dragon Troupes Based on Geometric Models

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Abstract. This paper focuses on the mathematical modeling and turning optimization of the intangible cultural heritage art "Bench Dragon," with an emphasis on the kinematic analysis of the dragon's head and body, collision detection within the formation, determination of the maximum travel time, and optimization of the minimum turning radius. By integrating methods from analytical geometry, numerical solutions of differential equations, optimization models, and algorithms, a multidisciplinary analytical framework was constructed. Through dynamic simulation and iterative algorithms, the movement trajectory of the Bench Dragon's body was precisely reconstructed. A novel rectangular collision detection model was proposed to mathematically define the concept of "bench collision," and the maximum safe travel time for the formation was determined to be 414 seconds. Using the bisection method, the possible range for the minimum turning radius was hypothesized, and based on optimization algorithms, the interval length was gradually reduced to meet the precision requirement of 0.01, ultimately solving for the minimum turning radius of 0.423 meters. An extendable bisection method program was developed to achieve rapid solutions. The research results provide quantitative analysis tools for the digital preservation of traditional arts, and the algorithmic framework offers methodological reference value for the study of group motion systems.

Keywords: Bench Dragon, Equidistant spiral, Optimization algorithm, Geometric modeling.

1. Introduction

Currently, scholars both domestically and internationally have conducted extensive research on the motion laws of multi-segment object geometric models and their path optimization problems. The research directions mainly focus on two aspects: the first is algorithm and model optimization methods, and the second is applications oriented towards structure and control, providing theoretical support and methodological guidance for complex motion systems.Ben Asher Joseph Z[1] investigated the time-optimal trajectory planning problem for moving objects under non-slip and minimum turning radius constraints using optimal control theory and the Hodograph method. Changjun Wu[2] et al. proposed a two-stage collision detection method based on bounding boxes and primitives, offering real-time collision warnings and motion coordination optimization strategies for moving objects. Rongxin Song[3] et al. combined LSTM-Autoencoder with a multi-task learning attention mechanism to achieve dynamic collision avoidance decisions, optimizing object motion path planning and audience interaction strategies. Hua Yicun[4] et al. summarized multi-objective optimization algorithms for irregular Pareto fronts, providing diverse optimization solutions for moving objects with complex performance requirements. Hongbo Wang[5] et al. proposed a parameter-adaptive detection method, addressing the issue of traditional methods failing when model parameters change. Through dynamic modeling and state monitoring, the research improved the accuracy and timeliness of collision detection. Changlin Li[6] et al. proposed the FP-TTC model, which quickly predicts collision time using an attention mechanism and time series decoder, offering efficient collision detection and obstacle avoidance solutions. Jing Kangkang[7] et al. used Ivanov's non-smooth transformation and path integral methods to study the probability density evolution of collision vibration systems, providing numerical analysis methods for collision problems. Zheng

Jin[8] et al. summarized the formation methods and characteristics of equiangular spirals, offering mathematical theoretical support for the design of complex motion paths for objects. Kusuma Sudhanva[9] et al. proposed a velocity-based arc length method, simplifying the solution process of extended equations and providing algorithmic support for precise tracking of motion paths. He Zhiping[10] et al. proposed an adaptive sliding mode control strategy, achieving cooperative trajectory tracking for multiple unmanned surface vehicles and providing control strategies for multi-segment cooperative motion. The aforementioned research by domestic and international scholars has achieved significant theoretical results in algorithm optimization, structural design, path planning, and dynamic control. Based on their work, this paper will further explore methods such as analytic geometry, numerical solutions of differential equations, collision avoidance strategies, and algorithms, providing more comprehensive solutions for the motion optimization of complex systems like the "Bench Dragon."

Compared to the two-stage collision detection method based on bounding boxes and primitives proposed by Changjun Wu[2], this paper introduces rectangular geometric constraints, strictly defining the criteria for "bench collision" from a mathematical perspective, and elevates the detection dimension from a two-dimensional plane to a three-dimensional rigid body motion analysis, better aligning with the actual physical structure of the Bench Dragon. This paper also iteratively optimizes the minimum turning radius, using the bisection method instead of the gradient descent method recommended by [3], to optimize the minimum pitch. By establishing a mapping relationship model between pitch and turning force, it addresses the estimation problems caused by traditional methods ignoring the dynamic coupling between benches. Compared to the rigid body constraint model of [4], this paper adds dynamic constraints for the transmission of forces between benches.

2. Methods

2.1. Kinematic Investigation of Dragon Head-Body Dynamics in Bench Performance

The first part of the study employs a helical motion model to analyze the movement of the bench dragon. The bench dragon is composed of 223 sections (the dragon head is 341 cm, the dragon body and tail are each 220 cm, with a single section width of 30 cm). The dragon head moves at a constant speed of 1 m/s along a helical path with a pitch of 55 cm. A trajectory model is established using polar coordinate equations, and the coordinates of the front handle of the dragon head are obtained by solving differential equations. The positions of subsequent sections are determined through geometric recursion. Based on the fixed section length characteristic, motion continuity is achieved using an adjacent section position recursive algorithm. Speed analysis reveals that the dragon head moves at a constant speed, while the speeds of the dragon body and tail dynamically change with the configuration. The instantaneous speed of any section can be calculated through position derivatives, revealing the transmission mechanism of the dragon head's movement on the overall dynamic characteristics.

According to the knowledge of analytic geometry, the polar coordinate equation of an equiangular spiral is:

$$\rho = \frac{p}{2\pi}\theta \tag{1}$$

Where p is the pitch of the spiral, which is 0.55m in Problem 1.

2.1.1. Determine the position of the faucet at each moment.

$$\frac{ds}{dt} = v_A = 1\mathrm{m/s} \tag{2}$$

Where ds is the arc differential of the equiangular spiral, v_A is the linear velocity of point A. According to the knowledge of differential calculus, in polar coordinates:

$$ds = \sqrt{\rho'^2 + \rho^2}\, d\theta \tag{3}$$

$$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} \tag{4}$$

By combining equations (1), (3), and (4), and considering the initial position of the faucet:

$$\theta(0) = 32\pi \tag{5}$$

The initial value problem for the differential equation of motion of the handle of the bench dragon head:

$$\begin{cases} \sqrt{\rho'^2 + \rho^2}\, \dfrac{d\theta}{dt} = v_A \\ t = 0 : \theta = 32\pi \end{cases} \tag{6}$$

By utilizing the transformation relationship between Cartesian and polar coordinates, solving equation (6) allows us to determine the positional coordinates of the bench dragon head at any given moment in time.

2.1.2. Determination of Handle Center Coordinates on the Bench Dragon Body

$$\begin{cases} l_1 = 2.86\text{m} \\ l_2 = 1.65\text{m} \end{cases} \tag{7}$$

Here, l1 represents the distance between the front and rear holes of the bench to which the dragon head belongs, and l2 represents the distance between the front and rear holes of the bench to which the dragon body and dragon tail belong. We employ an iterative method to sequentially determine the position of each subsequent hole based on the position of the previous hole. Taking the determination of the rear handle center position based on the front handle center position of the dragon head as an example:

$$\begin{cases} \left(x_1 - x_0\right)^2 + \left(y_1 - y_0\right)^2 = l_1^2 \\ \sqrt{x_1^2 + y_1^2} = \dfrac{0.55}{2\pi} \arctan \dfrac{y_1}{x_1} \end{cases} \tag{8}$$

Where the x is abscissa (x-coordinate) of the equiangular spiral(the ordinate (y-coordinate) of the equiangular spiral),and (x0, y0) represents the coordinates of the faucet at a certain moment. By solving this system of equations, we can determine the coordinates (x1, y1) of the center of the next handle at that same moment. Similarly, the position coordinates of the centers of the remaining handles at any given moment can be obtained.

2.1.3. Determine the magnitude of the velocity of each handle's center at various moments.

We adopt the method where the average velocity is considered as the instantaneous velocity. Let x(n) be the horizontal coordinate of a point at time t=n, and y(n) be the vertical coordinate of the same point at time t=n. Then, the magnitude of the velocity of the point at time t=n is expressed as:

$$v(n) = \sqrt{\left(\frac{x(n+1) - x(n)}{1s}\right)^2 + \left(\frac{y(n+1) - y(n)}{1s}\right)^2} \tag{9}$$

2.2. Collision Detection and Maximum Travel Time Determination.

The core of the second part is to address the following question: How to quantify "collision"? Or, in other words, how to mathematically characterize the collision between benches. According to common sense, for two rectangular benches to collide, it usually happens when a corner of one bench "hits" a side of the other bench. From a geometric perspective, this means that a vertex of one rectangle coincides with a side of another rectangle, as shown in Figure 1:

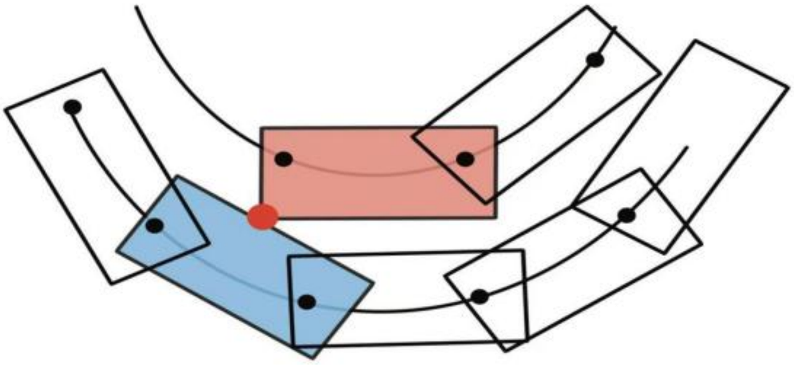

Figure 1. Rectangular Local Collision Diagram

Therefore, if we can determine the coordinates of the four vertices of the rectangle (bench) to which the dragon's head belongs, we only need to check whether any of these four points fall inside the rectangles of the rest of the dragon's body to determine if a collision has occurred between the dragon's head and the body. We propose the following criterion for this check: if the distance between a vertex of the dragon's head and the line connecting the centers of the front and rear handles of the dragon's body (the central axis of the bench) is less than half the width of the bench, i.e., 0.15m, a collision is considered to have occurred. This criterion is both mathematically and intuitively obvious. The specific solution process is as follows

2.2.1. Determine the coordinates of the four vertices of the rectangle (bench)

Take t=n (n=0,1,2,...) as an example to identify the rectangle (bench) to which the bench head belongs. Where ρ_n^A is the polar radius of point A at the nth second,the same as B.

$$A\left(\rho_n^A,\theta_n^A\right)\overset{Cartesian\ coordinates}{\Rightarrow}\left(x_n^A,\theta_n^A\right) \tag{10}$$

$$B\left(\rho_n^B,\theta_n^B\right)\overset{Cartesian\ coordinates}{\Rightarrow}\left(x_n^B,\theta_n^B\right) \tag{11}$$

Here, point A represents the coordinates of the front handle center of the bench dragon head, and point B represents the coordinates of the rear handle center of the bench dragon head. The subscript n denotes the moment t = n .

Next, we solve for the direction vectors of points A and B:

$$\overrightarrow{S_{AB}}=\left(\frac{x_n^A-x_n^B}{\sqrt{\left(x_n^A-x_n^B\right)^2+\left(y_n^A-y_n^B\right)^2}},\frac{y_n^A-y_n^B}{\sqrt{\left(x_n^A-x_n^B\right)^2+\left(y_n^A-y_n^B\right)^2}}\right) \tag{12}$$

Thus, it can be concluded that the angle α between line segment AB and the x-axis satisfies:

$$\begin{cases}\cos\alpha=\dfrac{x_n^A-x_n^B}{\sqrt{\left(x_n^A-x_n^B\right)^2+\left(y_n^A-y_n^B\right)^2}}\\ \sin\alpha=\dfrac{y_n^A-y_n^B}{\sqrt{\left(x_n^A-x_n^B\right)^2+\left(y_n^A-y_n^B\right)^2}}\end{cases} \tag{13}$$

Define the coordinate system rotation matrix.:

$$\mathbf{M}(\alpha)=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix} \tag{14}$$

$$\begin{bmatrix} x_n^{A'} \\ y_n^{A'} \end{bmatrix} = \mathbf{M}(\alpha) \begin{bmatrix} x_n^{A} \\ y_n^{A} \end{bmatrix} \tag{15}$$

The coordinates of the rotated point A' can thus be obtained. Using the coordinates of A', the coordinates of the four vertices of the rotated rectangle (bench) to which the dragon head belongs can be determined, namely:

$$a'\left(x_n^{A'} + 0.275, y_n^{A} + 0.15\right) = \left(x_n^{a'}, y_n^{a'}\right) \tag{16}$$

$$b'\left(x_n^{A'} + 0.275, y_n^{A} - 0.15\right) = \left(x_n^{b'}, y_n^{b'}\right) \tag{17}$$

$$c'\left(x_n^{A'} - 3.135, y_n^{A} - 0.15\right) = \left(x_n^{c'}, y_n^{c'}\right) \tag{18}$$

$$d'\left(x_n^{A'} + 3.135, y_n^{A} + 0.15\right) = \left(x_n^{d'}, y_n^{d'}\right) \tag{19}$$

Apply a rotation in the opposite direction to A' :

$$\begin{bmatrix} x_n^{a} \\ y_n^{a} \end{bmatrix} = \mathbf{M}^{-1}(\alpha) \begin{bmatrix} x_n^{a'} \\ y_n^{a'} \end{bmatrix} \tag{20}$$

Thus, the coordinates of one of the vertices a of the rectangle (bench) to which the dragon head belongs can be obtained. The coordinates of vertices b , c , and d can be determined in the same manner.

2.2.2. Establish the equations of the lines

Determine the equations of the lines to which the center connections of the handles on the bench dragon body belong. Assuming C and D are the centers of the front and rear handles of a certain section of the bench dragon body:

$$\begin{gathered} C\left(x_n^{C}, y_n^{C}\right) \\ D\left(x_n^{D}, y_n^{D}\right) \end{gathered} \tag{21}$$

Then, the equation of the line passing through C and D can be established as:

$$y = kx + l \tag{22}$$

$$\begin{cases} k = \dfrac{y_n^{D} - y_n^{C}}{x_n^{D} - x_n^{C}} \\ l = y_n^{D} - \dfrac{\left(y_n^{D} - y_n^{C}\right) x_n^{D}}{x_n^{D} - x_n^{C}} \end{cases} \tag{23}$$

2.2.3. Establish collision conditions

If the rectangle (bench) to which the dragon head belongs does not collide with the bench to which C and D belong, then the distances from points a, b , c , and d to the line (22) must be no less than 0.15 meters, that is:

$$p = \frac{m + 0.55}{2} \tag{24}$$

2.3. Optimize the minimum pitch

For the "Bench Dragon" team's turning space, refine the minimum pitch parameter.The objective of the third section is to determine the minimum pitch required for the dragon dance team to move along a spiral path, ensuring that the dragon head stops within a circular area of radius 4.5 meters.

Initially, the pitch range is set between 0 and 0.55 meters. Then, the pitch range is progressively narrowed using the bisection method: in each iteration, the midpoint of the current pitch range is calculated, and the final position of the dragon head at this pitch is tested to see if it lies within the circular area. If the dragon head can enter the circular area, the lower bound of the pitch range is updated to the current pitch; otherwise, the upper bound is updated. This process is repeated until the pitch range is reduced to less than 0.01 meters. Finally, the left endpoint of the pitch range is taken as the minimum pitch value, ensuring that the dragon head can stop within the specified circular area. Through the bisection method, the pitch range is gradually narrowed, ultimately determining the minimum pitch that satisfies the condition.

The process of solving for the minimum pitch employs the concept of the bisection method, and the specific operational procedure is as follows:

When p=0.55m, the bench dragon head can precisely reach the position (2.1054, 0.7328) without any collision occurring. In accordance with the problem's requirements, the bench dragon head needs to navigate into a turning circular area centered at the origin of the spiral (the center point) with a radius of 4.5 meters, that is to say:

$$x^2 + y^2 \le 4.5^2 \tag{25}$$

It is evident that when p=0.55m, the final stopping position of the bench dragon head lies within the circular area. Therefore, we proceed to reduce the value of p, specifically setting p=m where 0<m<0.55, to ensure that under this pitch, the bench dragon head ultimately halts outside the circular region. At this juncture, we adopt:

$$p = \frac{m + 0.55}{2} \tag{26}$$

Under this pitch, we examine whether the stopping position of the bench dragon head lies within the circular area:

2.3.1. Equation applicable within the circular region

$$p = \frac{m + \frac{m + 0.55}{2}}{2} \tag{27}$$

Under this pitch, we verify whether the stopping position of the bench dragon head lies within the circular area.

2.3.2. Equation applicable outside the circular area

$$p = \frac{0.55 + \frac{m + 0.55}{2}}{2} \cdot \tag{28}$$

Under this pitch, we verify whether the stopping position of the bench dragon head lies within the circular area.Following this approach, employing the bisection method, we continuously narrow down the interval length ε (the precision value of the bisection method) of the pitch p until:

$$\varepsilon < 0.01$$

Then, we take the pitch value corresponding to the left endpoint of this interval as the minimum pitch. The above constitutes the mathematical model for problem three.

3. Results

3.1. Kinematic analysis results

Present the kinematic analysis results for the dragon head and body in the "Bench Dragon" performance.

(1) Utilizing Python's numerical algorithms for solving ordinary differential equations, we can obtain the coordinates of the faucet's front handle center from 0s to 300s. Although a numerical solution method is employed, the deviation from the analytical solution is negligible.

(2) This problem involves solving a system of nonlinear equations using the scipy library, which allows us to find the numerical optimal solution for equation (8), thereby determining the position coordinates of the faucet body and tail at various moments.

(3) Based on the results from (1) and (2), writing a program to iteratively calculate the magnitude of the velocity of each handle at various moments is straightforward. The results are as shown in Table 1 and Table 2.

Table 1. Coordinates of the Bandeng Dragon's Head, Body, and Tail at Each Momen

Part and Coordinate Axis	0 s	60 s	120 s	180 s	240 s	300 s
Dragon's headx (m)	8.800000	5.799165	-4.088683	-2.924664	2.631660	4.419575
Dragon's heady (m)	0.000000	-5.771135	-6.301955	6.112949	-5.339255	2.321818
The1st section of the dragon's body. x (m)	8.366286	7.452784	-1.460651	-5.200249	4.835382	2.480084
The1st section of the dragon's body.y (m)	2.819001	-3.448764	-7.402721	4.402417	-3.543280	4.390431
The 51st section of the dragon's body. x (m)	-9.512018	-8.671372	-5.494043	2.996475	5.952480	-6.289513
The 51st section of the dragon's body. y (m)	1.385148	2.588941	6.419419	7.204716	-3.869304	0.587078
The 101st section of the dragon's body. x (m)	2.841064	5.616358	5.276274	1.741172	-4.998817	-6.133162
The 101st section of the dragon's bodyy (m)	-9.938777	-8.050273	-7.616497	-8.503753	-6.314817	4.092999
The 151st section of the dragon's bodyx (m)	10.877538	6.770928	2.521627	1.202159	3.103582	7.162351
The 151st section of the dragon's bodyy (m)	1.727014	8.059672	9.692577	9.399804	8.348097	4.186814
The 201st section of the dragon's bodyx (m)	4.671434	-6.503727	-10.603486	-9.379430	-7.571611	-7.614541
The 201st section of the dragon's bodyy (m)	10.673920	9.108100	1.524393	-4.035083	-6.037306	-5.030362
Dragon's Tail (Rear)x (m)	-5.427245	7.245898	10.985131	7.553761	3.426761	2.074396

Table 2. Coordinates of the Dragon's Head, Body, and Tail at Each Moment

Part and Coordinate Axis	0 s	60 s	120 s	180 s	240 s	300 s
Dragon's head (m/s)	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
The 1st section of the dragon's bodyx (m) (m/s)	0.999454	0.999350	1.000237	0.998174	0.998063	0.998329
The 51st section of the dragon's bodyx (m) (m/s)	0.999385	0.999260	1.000115	0.997997	0.997785	0.997833
The 101st section of the dragon's bodyx (m) (m/s)	0.999334	0.999197	1.000033	0.997888	0.997632	0.997601

The 151 st section of the dragon's bodyx (m) (m/s)	0.999296	0.999150	0.999975	0.997814	0.997535	0.997467
The 201st section of the dragon's body (m/s)	0.999265	0.999114	0.999932	0.997760	0.997467	0.997380
Dragon's Tail(Rear)(m/s)	0.999254	0.999101	0.999916	0.997741	0.997444	0.997350

3.2. Collision detection and travel time analysis results

Summarize the collision detection and maximum travel time analysis results for the "Bench Dragon" team.Based on the results obtained from the first part and combined with the model from the second part, the coordinates of the four vertices of the rectangle (bench) to which the dragon head belongs at any given moment can be determined. At each moment, the distances between these four vertices and the lines to which the center connections of the handles of the remaining dragon body and tail belong are checked. Using the discrimination method from (23), by iterating through all moments, the position coordinates of the dragon head at the exact moment of collision can be obtained.

In practice, during the actual execution of the code, there is no need to check the first section of the dragon body because it is always hinged to the dragon head. Additionally, it is unnecessary to check all sections of the dragon body since the dragon head will only collide with the adjacent sections around it. Therefore, we examined the collision conditions for sections 3 to 15 of the dragon body, checking every 1 second.

At 414 seconds, the bench dragon head experienced a collision. At this moment, the coordinates and velocities of the handles of the bench dragon head, bench dragon body, and bench dragon tail are shown in Fig2 below.

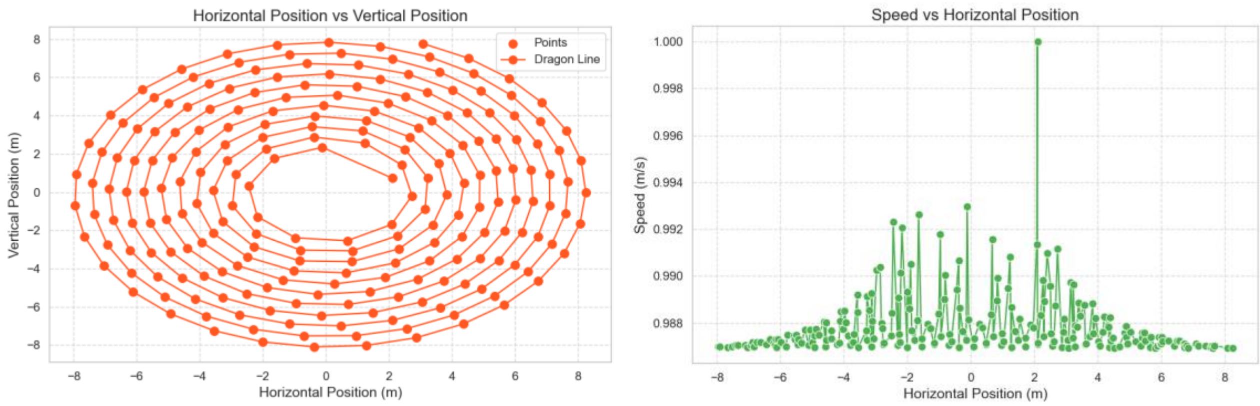

Figure 2. Horizontal and Vertical Position Diagram and Velocity-Horizontal Position Relationship Diagram.

3.3. Minimum pitch optimization results

The program-generated optimization results indicate a minimum pitch of 0.423 meters in the "Bench Dragon" team's turning space.

4. Conclusions

This paper focuses on the motion characteristics and path optimization of the traditional "Bench Dragon" art. By integrating geometric analysis, differential equations, and numerical algorithms, it comprehensively analyzes the motion features of the Bench Dragon. The model, based on spiral curves and combined with the Python numerical computation framework, achieves dynamic simulation and collision detection of the Bench Dragon's trajectory, with clear geometric-physical relationships and high computational efficiency. A collision detection mechanism based on analytic geometry is established, determining a maximum safe travel time of 414 seconds. The model optimizes the maximum turning path, demonstrating theoretical rigor and engineering scalability. It can be extended to motion analysis of multi-joint robots, multi-section trains, and other chain-like structures, offering both cultural value and engineering application potential. However, the current study has the following limitations: (1) Computational complexity increases when handling U-turns and complex path planning due to the complexity of polar coordinates. (2) The detection of U-turns and collisions is overly idealized. Collision analysis does not consider other interactions between benches, such as relative motion directions. In practical applications, bench detection requires more complex three-dimensional motion analysis. The U-turn process assumes smooth steering, and the model does not address optimization on non-smooth paths. Future improvements include: (1) Optimizing computational complexity by introducing efficient algorithms (e.g., Fast Fourier Transform or parallel computing) to simplify polar coordinate calculations and enhance complex path planning efficiency. (2) Considering relative motion directions between benches and introducing three-dimensional motion analysis and dynamic collision models to improve detection accuracy. (3) Enhancing the U-turn model by optimizing steering strategies on non-smooth paths and integrating real-time path optimization algorithms to ensure smoother and more stable U-turns.

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