# An Analysis of Approaches to Goldbach's and De Polignac's Conjectures and Their Interconnections

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**Abstract.** Goldbach's Conjecture and de Polignac's Conjecture remain two of the most persistent unsolved problems in the additive properties of primes and have been so for nearly two centuries. This paper attempts a thorough exercise on major analytical methods currently available for adjudging these conjectures. Analytically, this paper discusses two main theorems that have been used to address Goldbach's Conjecture: that of Chen Jingrun and the Hardy-Littlewood Circle Method. Similarly, the discussion provides the Goldston-Pintz-Yıldırım sieving, which relates to bounded prime gaps, and critically surveys a specific purported proof of de Polignac's Conjecture from 2014. It also reports the results of a computational verification done to prove the conjecture for evens up to 10,000, together with an associated property. This turns out to be with defects in the earlier purported proof. The paper ends by reiterating the closeness of the Conjectures-in demonstrating their deep entanglement with the prime number distribution function-as well as the difficulty inherent in connecting two apparently disparate behaviors of the same object, even though both are tied to their additive disposition.

**Keywords:** Goldbach's Conjecture; De Polignac's Conjecture; Additive Number Theory; Sieve Methods; Prime Gaps.

## 1. Introduction

Prime numbers play a basic role in mathematics. They are natural numbers greater than 1 which have no positive divisors other than 1 and themselves. Since about 300BC Euclid proved their infinitude, mathematicians have engaged the problems of their distribution and properties. The greats like Fermat, Euler, Goldbach, and many other mathematicians worked on the problems of prime numbers to relatively recent theorems and tough conjectures such as the Riemann Hypothesis, Goldbach's Conjecture, and Fermat's Little Theorem. Although the primes are at the base of multiplicative structures, a coherent account of their multiplicative and additive behavior is a task of astounding difficulty that is mostly fruitless.

In the domain of number theory, much has been attained; however, many queries regarding prime numbers remain unsolved. This article takes up these two well-known problems: the conjecture of Goldbach and the conjecture of de Polignac.

The paper sets forth the following aims: to review major analytical approaches elaborated for every conjecture, to conduct a critical analysis of a particular alleged proof of de Polignac's Conjecture published in 2014 [1], to check computationally within a small scope the similarities and possible links between these two additive problems with primes. This paper is outlined as follows: Section 2 formulates the background and the conjectures' explicit statements. Section 3 discusses a review of the approaches toward Goldbach's Conjecture, specifically the sieve methods that lead up to Chen's Theorem and The Hardy-Littlewood Circle Method. Section 4 checks approach toward de Polignac's Conjecture: Goldston-Pintz-Yıldırım (GPY) sieve for bounded gaps and Marshall's attempt at proof. Section 5 gives computational verification results. Section 6 discusses the interconnections and difficulties between these conjectures. Section 7 makes the final concluding remarks.

## 2. Background and Preliminaries

Key terms: Prime Number, A natural number $p>1$ whose only positive divisors are 1 and p. Semiprime: The product of two prime numbers (not necessarily distinct primes). Goldbach's

Conjection: Strong Form In a letter sent by Christian Goldbach to Leonhard Euler in 1742 and later redefined by Euler, it was declared that "Every even integer greater than 2 is the sum of two primes" [2]. Examples include those noted in Eq. (1). The initial problem was about the summation of three primes, taking 1 as a prime, which gave the 'Weak' Goldbach Conjecture, now confirmed by Helfgott.

$$\begin{aligned} 4 &= 2 + 2 \\ 48 &= 37 + 11 \\ 102 &= 23 + 79 \end{aligned} \tag{1}$$

It was in 1849 that Alphonse de Polignac made what has come to be known as De Polignac's Conjecture. It says that "For any positive even integer n (often written as 2k), there are infinitely many pairs of prime numbers (p, p+n) [3]" as suggested by Eq. (2). The case n=2 is the famous Twin Prime Conjecture. This conjecture would mean that there are infinitely many prime gaps of any given even size.

$$2k = a-b \tag{2}$$

where a, b are prime numbers and $k \in Z+$.

# 3. Approaches to Goldbach's Conjecture

## 3.1. Sieve Methods and Chen's Theorem

Sieve methods are a powerful class of techniques in number theory used to estimate the size of sifted sets of integers. They are central to progress on Goldbach's conjecture.

### 3.1.1 Sieving Basics

The fundamental idea, originating from the Sieve of Eratosthenes, is to remove numbers with small prime factors from a set. The Möbius function μ(n) is defined according to Eq. (3).

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \text{when } n \text{ has square factor} \\ (-1)^s & n = P_1 \times P_2 \times P_3 \ldots \times P_s \end{cases} \tag{3}$$

It possesses an important property, denoted as MP (Möbius function's property) in the original text and shown in Eq. (4),

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases} \tag{4}$$

where the sum of μ(d) over the divisors d of n is 1 if n=1 and 0 otherwise. The function P(z) is defined in Eq. (5) as the product of all prime numbers less than or equal to z.

$$P(z) = \prod_{\text{i is prime number} \le \lfloor z \rfloor} i \tag{5}$$

The sieving function S (A, P, z) is defined by Eq. (6) in simple words as counting the elements *a* in a set A, coprime to P(z) having no common prime factors ≤ z.

$$S(A, P, Z) = \sum_{\substack{a \in A \\ ((a, P(z)) = 1}} 1 \tag{6}$$

Using the property from Eq. (4), S (A, P, z) gets written in terms of μ(d) and counts of elements in A that are divisible by $d(|A_d|)$, as shown in steps from Eq. (7) to (9).

$$\begin{aligned} S(A, P, Z) &= \sum_{\substack{a \in A \\ ((a, P(z)) = 1}} 1 \\ &= \sum_{a \in A} \sum_{d|(a, P(Z))} \mu(d) \\ &= \sum_{a \in A} \sum_{\substack{d|a \\ d|P(Z)}} \mu(d) \end{aligned} \tag{7}$$

$$\sum_{a\in A}\sum_{\substack{d|a\\ d|P(Z)}}\mu(d)=\sum_{d|P(Z)}\mu(d)\sum_{\substack{a\in A\\ d|a}}1 \tag{8}$$

$$\begin{aligned}\sum_{d|P(Z)}\mu(d)\sum_{\substack{a\in A\\ d|a}}1&=\sum_{d|P(Z)}\mu(d)|A_d|\\ &=\sum_{d|P(Z)}\mu(d)\left(\frac{X-Z}{d}+\phi\right)\\ &=(X-Z)\sum_{d|P(Z)}\frac{\mu(d)}{d}+\sum_{d|P(Z)}\mu(d)\phi\end{aligned} \tag{9}$$

Typically, this turns out to be a main term plus an error term. Eq. (10) gives an idea for the count of multiples of d in an interval.

$$\left\lfloor\frac{x+y}{d}\right\rfloor-\left\lfloor\frac{x}{d}\right\rfloor=\frac{y}{d}+\phi|\phi|\leq 1 \tag{10}$$

What is given after Eq. (9) is an upper bound, as is usually defined for partially ordered sets.

The subsequent derivation leading to the upper bound for π(X) in Eq. (11) uses these sieve concepts.

$$\pi(X)\leq Z+S(A,P,Z)<Z+\frac{X}{\ln Z}+2^Z \tag{11}$$

For applying the above to the Goldbach conjecture (N = p1 + p2), consider the set A = {N - p: p is prime, p < N}. It is what will have set A containing at least one prime number (p2) for N sufficiently large. Sieve methods have aimed at finding a positive lower bound for the number of primes (or almost primes) in A.

### 3.1.2 Chen's Theorem

Based on sieve methods (e.g., the Jurkat-Richert sieve, as in work by Buchstab and others), Chen Jingrun, a Chinese mathematician, achieved the most significant result to date in proving Goldbach’s conjecture. In 1973, Chen established that: Every sufficiently large even integer N can be written as the sum of a prime and a number that is either prime or semiprime (i.e., the product of two primes) [4]. This result is indicated in the form N = p + q × m displayed in Eq. (12).

$$N=p+q\times m \tag{12}$$

This can be written as N = p + P2. Chen used strong weighted sieves in proving the above and used results such as the Bombieri-Vinogradov theorem [5] and the Richert theorem [6]. What he found was that the number of ways to represent N in the "1+2" manner, which is $P_x(\alpha,\beta)$ by notation of Eq. (13), is in fact greater than that of a positive manner, as can be seen from the inequality in Eq. (14) where the constant $C_x$ is given by Eq. (15) involving that from Eq. (16).

$$P_x(\alpha,\beta)=p_1\times p_2\times p_3\times\ldots\times p_\alpha+p_a\times p_b\times p_c\times\ldots\times p_\beta \tag{13}$$

$$P_z(1,2)\geq\frac{0.67xC_x}{\log x^2} \tag{14}$$

$$C_x=\prod_{\substack{p|x\\ p>2}}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) \tag{15}$$

$$\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)=0.660161\ldots \tag{16}$$

The theorem of Chen does not, in its entirely proved form, supersede all others— it does not supersede all others— but it does come to be the strongest such theorem of its kind.

## 3.2. The Hardy-Littlewood Circle Method

The Circle Method amounts to a powerful analytic number theory technique, introduced by G.H. Hardy and J.E. Littlewood (and later improved by Vinogradov) for attacking additive problems such as Goldbach's Conjecture and Waring's problem [7].

The main notion is to represent the number of solutions to an additive equation as a complex integral over the unit circle. For the conjecture of Goldbach, specifically representing 2n as p1 + p2 as in Eq. (17), the number of solutions R(n) can be written via the integral in Eq. (18). This will be based on the orthogonality property of complex exponentials as given in Eq. (19).

$$2n = p1 + p2 \quad (17)$$

$$R(n) = \int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i\alpha p}\right)^2 e^{-2\pi i(2n)\alpha} d\alpha \quad (18)$$

$$\int_0^1 e^{2\pi i\alpha x} dx = \begin{cases} 1, \text{ if } \alpha = 0 \\ 0, \text{ if } \alpha \neq 0 \end{cases} \quad (19)$$

The integral is evaluated by splitting the interval [0,1] into two parts. The Major Arcs (M) comprise small intervals around rational numbers of the form α = a/q with small denominators of q. This will be the main source of primes reflecting the distribution of primes in arithmetic progressions on R(n). The Minor Arcs (m) take up the remainder of the interval [0,1]—their contribution being smaller (an error term) is expected to take place [7].

Hardy and Littlewood demonstrated that the for major arcs, if expected asymptotic formula as Eq. (20) having R(n) > 0 for large even N (or 2n) is to be obtained.

$$\int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i\alpha p}\right)^2 e^{-2\pi i(2n)\alpha} d\alpha \sim c \prod_{\substack{p|2n \\ p>2}} \frac{p-1}{p-2} \cdot \frac{n}{ln^2 n} \quad (20)$$

The major difficulty lies in proving that, for the minor arcs, if expected asymptotic formula is significantly smaller than the main term for major arcs essentially to prove that the relation in Eq. (21) does not hold.

$$\int_0^1 \left(\sum_{p\leq 2n} e^{2\pi i\alpha p}\right)^2 e^{-2\pi i(2n)\alpha} d\alpha \sim 0\left(\frac{n}{ln^n}\right) \quad (21)$$

Hardy and Littlewood showed that, assuming the Generalized Riemann Hypothesis (GRH), the minor arc contribution is actually less, thus showing that every sufficiently large odd integer is the sum of three primes (Weak Goldbach) and that almost all even integers are the sum of two primes. This was later achieved by Vinogradov. Unconditionally establishing the Strong Goldbach Conjecture by the Circle Method is still largely inaccessible because of the problems in bounding the minor arcs.

# 4. Approaches to de Polignac's Conjecture

## 4.1. Bounded Gaps Between Primes and the GPY Sieve

The conjecture of De Polignac implies infinitely many prime pairs (p, p + 2k), as in Eq. (2), it means there are primes that are relatively close to each other up to any distance. Mostly, the study is focused on its related bounded gaps between primes [8].

Let $d_n = p_{n+1} - p_n$ be the gap between consecutive primes. The average gap size behaves like log $p_n$ as suggested by Eq. (22) since the Prime Number Theorem (see in Figure 1) [8].

$$\frac{x}{\pi(x)} \sim \frac{x}{\frac{x}{\log x}} \sim \log x \quad (22)$$

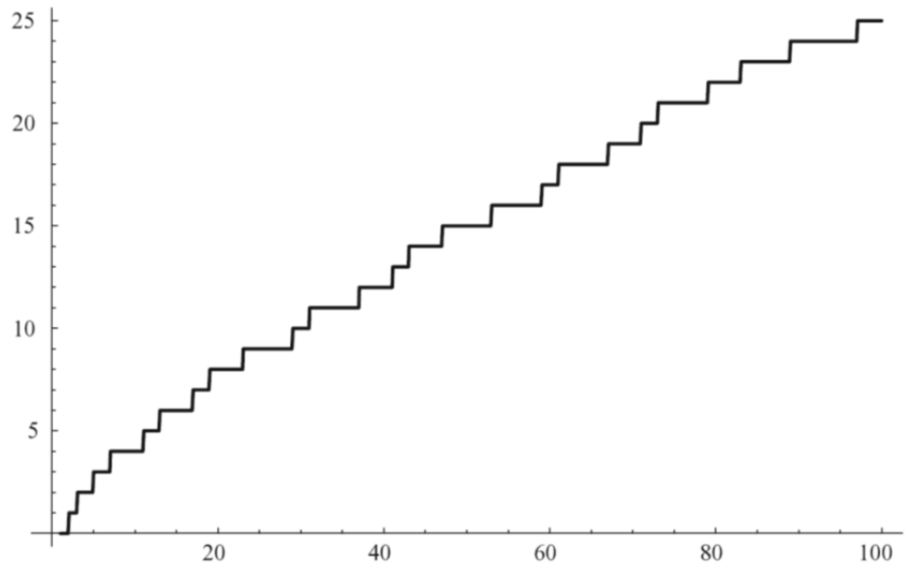


**Figure 1.** The graph of an(x) for1 <x≤ 100 [8].

A major goal has been to prove that gaps remain bounded infinitely always, meaning the limit inferior of the normalized gaps, Δ as defined in Eq. (23), is finite (specifically less than 1, ideally 0).

$$\Delta := \liminf_{n\to\infty} \left(\frac{p_{n+1}-p_n}{\log p_n}\right) \tag{23}$$

Proving this would not directly prove de Polignac's conjecture for all 2k, but would be a monumental advance.

Progress is based on the distribution of primes in arithmetic progressions, described by π (x; q, a), the number of primes ≤ x congruent to a mod q. The Prime Number Theorem for Arithmetic Progressions gives the relationship as Eq. (24) [9] in terms of the logarithmic integral li(x) (Eq. 25) [10], and Euler's totient function φ(q).

$$\pi(x; q, a) \sim \frac{li(x)}{\phi(q)} \tag{24}$$

$$li(x) = \int_0^\infty \frac{1}{\ln (t)} dt \tag{25}$$

The Bombieri-Vinogradov Theorem gives main result on average quality in such approach over many moduli q as given in Eq. (26) for Q given in (27).

$$\sum_{q\leq Q} \max_{(a,q)=1} \left|\pi(x; q, a) - \frac{li(x)}{\phi(q)}\right| \leq C \frac{x}{(\log\, x)^A} \tag{26}$$

$$Q = \frac{x^{\frac{1}{2}}}{(\log\, x)^B} \tag{27}$$

It basically says that the primes are well-equalized in arithmetic progressions on average up to moduli q ≈ x^(1/2 - ε). The level of distribution ϑ gives the exponent such that the Prime Number Theorem (PNT) for Aps (Arithmetic Progressions) holds on average for q ≤ x^ϑ. Bombieri-Vinogradov gives ϑ = 1/2. The Elliott-Halberstam conjecture gives ϑ = 1.

The GPY Sieve method indicated that if there were ε > 1/2 + ϑ, there were infinitely many bounded prime gaps (Δ = 0).

While GPY didn't prove ϑ > 1/2 unconditionally, a variant of Bombieri-Vinogradov holds for ϑ > 1/2 when restricted to moduli with only small prime factors, as proved by Yitang Zhang [11]. This unconditionally proves lim inf $d_n$ < 70,000,000. Much later, James Maynard, Terence Tao, and a group labeled the Polymath Project whittled the bound down to where currently it stands, lim inf $d_n$ ≤ 246 [12]. Under the generalized Elliott-Halberstam hypothesis or any hypothesis of that nature, the bound would be ≤ 12 or ≤ 6 as mentioned in the version of the de Polignac conjecture [3] that was spotted in the introduction. The result referred to in Eq. (28) as a gap ≤ 16 would be either in some earlier work or in some specific condition within the context of any method related to GPY.

$$p_1 - p_2 \leq 16 \text{ for infinitely many prime numbers} \tag{28}$$

### 4.2. Critical Analysis of Marshall's (2014) Purported Proof

In a paper posted on viXra [1], Stephen Marshall proofs for several prime conjectures, including de Polignac's: it hinges on a criterion from Koninck and Mercier. p > 1 and d > 0 are integers such that p and p+d are both prime if and only if the expression for n given in Eq. (29) comes out to be an integer [1].

$$n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^d \times d!}{p+d}\right) + \frac{1}{p} + \frac{1}{p+d} \tag{29}$$

Marshall's Argument Summary: It proceeds by assuming there is a finite number of Polignac prime pairs (pi, pi + 2k) for fixed even 2k. A large number N is constructed as the product of all members of these finite pairs. The argument then attempts to use Eq. (29) via proof by contradiction to show there must exist another Polignac pair (q, q+2k) not in the original finite set, implying the set must be infinite. The transformation shown in Eq. (30) is part of this argument.

$$\begin{gathered} n = (p-1)!\left(\frac{1}{p} + \frac{(-1)^2 k \times d!}{p+2k}\right) + \frac{1}{p} + \frac{1}{p+2k} \\ \Longrightarrow (p+2k)np = (p-1)!\,(p + 2k + 2pk!) + 2(p+k) \end{gathered} \tag{30}$$

Critical issues with this argument are at least as much important. Firstly, with the condition on n, Eq. (29) writes the criterion very strictly in the form that n has to be an integer. The argument presented by Marshall [1] appears to consider cases where n might be a rational fraction. This contradicts the criterion's requirement that n must be an integer. Secondly, apparent circularity in the logic: the heart of the proof by contradiction attempts to use the properties of Eq. (29) which holds if ever p and p+d is prime to assert the existence of a new prime pair satisfying the conditions, seemingly without independent justification that there must be such a new pair outside the initially assumed finite set. The logic appears circular since the expression itself argues that the expression must create a new instance that satisfies its own preconditions. In this way, it becomes self-defeating since it undermines the validity of the proof that has been presented.

It is from these points that the alleged proof of de Polignac's Conjecture [1] seems logically unsound.

## 5. Computational Verification

While conjectures about infinitely many numbers cannot be proven computationally, code can verify them for a finite range, providing empirical support or potentially finding counterexamples (though none are expected for these conjectures).

### 5.1. Goldbach Verification

The coding for testing the first 10,000 even numbers will be written in Python in this paper. The running result shows that Goldbach’s conjecture works for the first 10,00 even numbers.

The code is written as shown in Figure 2:

```
def is_prime(a):
    for i in range(2,a-1):
        if a%i==0:
            return False
    return True
for number in range(4,10001,2):
    symbol=False
    for minus in range(2,number):
        if is_prime(minus)==True and is_prime(number-minus)==True:
            symbol=True
            print(number,"=", minus,"+", number-minus)
            break
    if symbol==False:
        print("Number", number, "is an exception")
```

**Figure 2.** Python code tests the Strong Goldbach Conjecture for even integers N from 4 to 10,000.

Result: Execution of the code verifies that Goldbach's conjecture holds for all even integers N from 4 to 10,000, inclusive.

### 5.2. De Polignac Related Verification

The coding for testing the first 10,000 even numbers will be written in Python in this paper. The running result shows that de Polignac's conjecture works for the first 10,00 even numbers. The code is written as shown in Figure 3.

```
def is_prime(a):
    for i in range(2,a-1):
        if a%i==0:
            return False
    return True
for number in range(6,10001,2):
    symbol=False
    for minus in range(2,number):
        if is_prime(minus)==True and is_prime(number-minus-2)==True:
            symbol=True
            print(number,"=", minus,"+", number-minus-2,"+",2)
            break
    if symbol==False:
        print("Number", number, "is an exception")
```

**Figure 3**. The python code for texting de Polignac's conjecture works for the first 10,00 even numbers.

## 6. Discussion

The review of approaches to Goldbach's and de Polignac's conjectures highlights several key points and interconnections. Avoiding the redundancy of similarities and connections, both conjectures are fundamentally about the additive properties of prime numbers and how they may bridge the gap between their multiplicative definition and how they behave under addition or subtraction. Goldbach wants to make even numbers by adding primes, while de Polignac explores the differences between primes. Moreover, advancement in both conjectures is deeply connected with understanding the distribution of primes, particularly in arithmetic progressions. This is explicitly revealed in the application of the Bombieri-Vinogradov theorem (Eq. 26) in Chen's proof [4] and that of GPY sieve and implicitly under the Circle Method major arc estimates as its conditional dependence on GRH. Potential mathematical transformations linking the two conjectures also tentatively suggest structural links between the problems.

Shared challenges also exist. Proving the conjectures is difficult due to the apparently random distribution of primes at local levels, although their global distribution is well described by the Prime Number Theorem. Present methods still do not have adequate control over the locations of primes to support the necessary additive structures universally. In the strong Goldbach conjecture, the Circle Method is useful due to the present restriction in obtaining unconditional bounds for the minor arcs. Although sieve methods are powerful, they often attain some parity problem level of inadequacy that bars a "1+1" result even with the boldest additional input. Work on bounded does not even begin to suggest a way, let alone give a whiff of a promise of help in finding a way, to tackle the problem of infinitely many lacunae of any given width.

The importance of partial results like Chen’s Theorem (Eq. 14) and the GPY/Zhang/Maynard results on bounded gaps should not be underestimated. It proves that primes do exhibit some additive regularity and accrues persuasive evidence for the veracity of the full conjectures. The analysis of Marshall’s paper [1] underscores the subtlety required in number theoretic proofs. Finally, some limitations apply to this review: it gives a high-level overview, and to fully understand the methods discussed one would need a large background in analytic number theory, not detailed here.

## 7. Conclusion

Goldbach's and de Polignac's Conjecture are some of the best-known expressions of the question regarding the additive properties of prime numbers. This paper tried to review the main analytical techniques that have been used in the study, including sieve methods -- up to Chen's groundbreaking theorem, which expresses large even numbers as the sum of a prime and almost prime, integral formulation of the Hardy-Littlewood Circle Method, and sieve techniques that eventually led to success in finding bounded prime gaps, on which de Polignac is relevant. We described a critical analysis of a purported proof of de Polignac's conjecture and, in that proof, showed the very serious logical inadequacies of an actual description based on the flimsy application of the underlying criterion. Checkings of computations up to N = 10,000 were carried out for both Goldbach's and de Polignac's Conjecture.

Both conjectures, Chen's theorem, and that of bounded prime gaps have not been solved though there has been great progress. It is not certain whether further depths into the fine distribution of prime numbers are needed or entirely new mathematical techniques. Because the two problems are somewhat similar, deriving their bases from the reflection of the prime in arithmetic progression through significant distribution theorems, it can be argued that an advance in one aspect may relate to the other. This, therefore, continues to inspire mathematical research, bringing to light some of the deep and rather unexpected connections in the theory of numbers.

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