Research On Speed Strategy of Freestyle Swimming Based on Lagrange Multiplier Method

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Abstract. This study investigates the optimal speed strategies for freestyle swimming events, focusing on balancing speed endurance and tactical approaches across different distances. Using the Lagrange multiplier method and Newton's iterative method, we developed a speed-distribution model to minimize total swimming time, considering constraints on speed, acceleration, and physical capacity. Additionally, a dynamic model based on game theory was established to describe athletes' dynamic speed distribution and strategic interactions, incorporating the Hamilton-Pontryagin principle and Nash equilibrium. Results show that short-distance events prioritize explosive power, while middle and long-distance events require balancing energy distribution and managing lactic acid accumulation. This research innovatively integrates mathematical modeling with sports physiology and game theory, providing a comprehensive framework to optimize freestyle swimming strategies and offering scientific recommendations for athletes to enhance performance.

Keywords: Lagrange Multiplier, Newton's Iterative Method, Hamilton-Pontryagin Principle, Freestyle Race, Strategy of Game.

1. Introduction

Current research on freestyle swimming strategies covers multiple aspects, including pacing strategies, technical optimization, and psychological adjustments. Many studies focus on pacing strategies in freestyle competitions, aiming to determine the optimal speed distribution over different race distances. For instance, research by Foster et al. (1993) and Thompson et al. (2003) has extensively explored the relationship between pacing and performance in endurance sports, including swimming. These studies suggest that even pacing is generally optimal for longer distances, while variable pacing strategies may be more effective in shorter events. Technical research in freestyle swimming primarily aims to optimize swimming techniques to improve efficiency, reduce water resistance, increase speed, and conserve energy. Studies by Sanders et al. (2001) and Toussaint et al. (2004) have investigated the biomechanics of freestyle swimming, focusing on stroke mechanics, body position, and propulsion efficiency. These studies have provided valuable insights into how technical adjustments can lead to performance improvements. Psychological strategies in swimming primarily address how adjusting mental states can enhance performance. Research by Raglin et al. (1990) and Hanin (2000) has explored the role of anxiety, confidence, and focus in athletic performance. These studies suggest that psychological factors can significantly impact performance, particularly in high-pressure competitive environments. However, most strategy recommendations are based on large sample statistical results, with insufficient consideration of individual differences. Psychological research often overlooks the impact of external factors on mental states, such as competition dynamics, opponent behavior, and environmental conditions. This paper aims to address these gaps by considering the effects of individual differences and external factors on competition strategies, offering more scientifically sound recommendations to help athletes swim faster.

2. The speed distribution model

2.1. The establishment of speed distribution model

2.1.1 Model objective

To minimize the total swimming time using integration, as shown in equation (1)

$$\mathrm{T} = \int_0^{\mathrm{D}} \frac{1}{\mathrm{v(s)}} \mathrm{ds} \tag{1}$$

In equation (1), D represents the total competition distance (50 meters, 100 meters, or 200 meters), and $v(s)$ is the swimmer's instantaneous speed at any given point along the swim distance.

2.1.2 Constraint conditions

Each athlete has different maximum speeds, abilities to maintain high speed during freestyle swimming, and acceleration capabilities. Therefore, the constraints imposed by these different factors need to be considered.

$$v(s) \le v_{max} \tag{2}$$

In equation (2), v_{max} is the athlete's absolute maximum speed.

$$\mathrm{E(s)} = \mathrm{E}_{\mathrm{total}} - \int_0^{\mathrm{s}} \mathrm{P(v(s))ds} \ge 0 \tag{3}$$

Lactic Acid Accumulation and Fatigue Constraints to Determine Energy Expenditure. In equation (3), $P(v(s))$ is the power consumption function corresponding to the speed $v(s)$,which typically satisfies $P(v) \propto v^3$.

$$\mathrm{a}_{\mathrm{min}} \le \mathrm{a(s)} \le \mathrm{a}_{\mathrm{max}} \tag{4}$$

The acceleration of an athlete, $a(s) = \frac{dv}{dt}$ cannot exceed the athlete's strength limit.

2.1.3 Match segmentation strategy

(1) Starting swimming phase (0~15 meters)

$$\mathrm{v(s)} = \mathrm{a} \times \mathrm{t} \tag{5}$$

In this phase, accelerate from a standstill to maximum speed. In equation (5), a is the maximum acceleration, and it is limited by the athlete's explosive power and efficiency of pushing water.

(2) Mid-range maintenance phase

$$\mathrm{v(s)} = \mathrm{v}_{\mathrm{max}} - \mathrm{k} \times \mathrm{s} \tag{6}$$

Maintain a state close to maximum speed while controlling physical energy consumption to prevent lactic acid from accumulating too quickly. In Equation (6), k represents the attenuation coefficient, which is related to the race distance and the athlete's endurance.

(3) Sprint phase

$$\mathrm{v(s)} = \mathrm{v}_{\mathrm{max}} - \mathrm{f(s)} \tag{7}$$

Athletes accelerate with all their remaining energy. In Equation (7), $f(s)$ ∝fatigue function.

2.1.4 Speed allocation strategy for different distances

(1) Simplified model for 50-meter freestyle

$$\mathrm{v(s)} \approx \mathrm{v}_{\mathrm{max}} \tag{8}$$

(2) 100-meter freestyle model form

$$\mathrm{v(s)} = \begin{cases} \mathrm{a} \times \mathrm{t}, & 0 \le \mathrm{s} \le 15 \\ \mathrm{v}_{\mathrm{max}} - \mathrm{k} \times \mathrm{s}, & 15 \le \mathrm{s} \le 85 \\ \mathrm{v}_{\mathrm{max}} - \mathrm{f(s)}, & 85 \le \mathrm{s} \le 100 \end{cases} \tag{9}$$

200-meter freestyle model form

$$v(s)=\begin{cases} a\times t, & 0\le s\le 15 \\ v_{max}-k\times s, & 15\le s\le 150 \\ v_{max}-f(s), & 150\le s\le 200 \end{cases} \tag{10}$$

2.2. The solution of the speed distribution mode

2.2.1 Solve the nonlinear equation about $v(s)$

The model one uses the Lagrange multiplier method to solve, transforming the constrained extremum problem of time in the model into an unconstrained extremum problem [1].

To optimize $\mathcal{L}$, we take the first derivative of the velocity $v(s)$ and set it to zero.

$$\frac{\partial\mathcal{L}}{\partial v(s)}=-\frac{1}{v(s)^2}+\lambda_1\times\frac{dP(v(s))}{dv}+\lambda_2 \tag{11}$$

Derivative of a power function: let $P(v)=k\times v^3$,then

$$\frac{dP(v)}{dv}=3\times k\times v^2 \tag{12}$$

Organize the equation: Substitute the derivative of the power function into the main equation, and obtain the following equation.

$$-\frac{1}{v(s)^2}+\lambda_1\times(3k\times v(s)^2)+\lambda_2=0 \tag{13}$$

The simplified equation is as follows

$$\lambda_1\times 3k\times v(s)^4+\lambda_2\times v(s)^2-1=0 \tag{14}$$

A nonlinear equation regarding $v(s)$ can be obtained, which will be solved using the Newton-Raphson method.

2.2.2 Add constraint conditions

$$\int_0^D P(v(s))ds\le E_{total} \tag{15}$$

Substitute the optimized solution for $v(s)$ into the power function $P(v)$ to verify if the energy constraint is satisfied. If not, adjust λ_1 and re-optimize [2].

Ensure that $v(s)\le v_{max}$. If not satisfied, a piecewise function can be introduced to limit the speed.

$$\frac{dv(s)}{ds}\in[a_{min},a_{max}] \tag{16}$$

By numerically testing the derivative of $v(s)$, adjust the initial conditions to satisfy it.

To solve the problem, the paper first initialize parameters, which include specifying the race distance and introducing the Lagrange multiplier. Subsequently, an iterative solution process is carried out using the Newton-Raphson method to optimize the Lagrangian function, with the aim of deriving the velocity distribution [3]. Finally, constraints are verified by computing the total energy consumption to ensure compliance with given limits and confirming that acceleration constraints are not exceeded. The paper applied the Lagrange multiplier method and the Newton-Raphson method, setting distances at 50 meters, 100 meters, and 200 meters, with acceleration factors for the initial phase being 2.5, 2.5, and 2.0 respectively, mid-section speed maintenance values at 2.0.The results are shown in Figure 1.

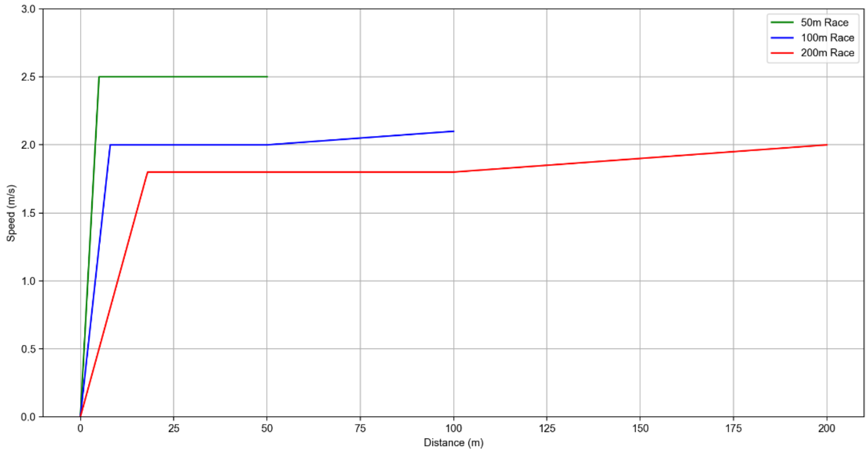

Figure 1. Optimal Speed Distribution for Different Distance Races

From the Figure 1, it can be seen that in the 50m freestyle race, the speed should quickly reach the absolute maximum and be maintained throughout, testing the athlete's explosive power more; in the 100m race, the speed rises rapidly at the beginning, and gradually stabilizes in the latter half without reaching the athlete's absolute maximum speed, which is to prevent lactic acid from accumulating too quickly and affecting the speed later on, and finally, the remaining physical energy is used to complete the sprint; the 200m race has a relatively even speed, testing the athlete's endurance more, and the final sprint is made using the remaining physical energy [4]. The above methods can reasonably arrange the speed to achieve the best results.

2.2.3 Model validation and evaluation

To further test and evaluate the model, we applied the dynamic programming approach to the analysis of speed strategies in swimming competitions, which highly aligns with the conclusions about different distance freestyle races mentioned above [5].

3. Dynamic models based on game theory

3.1. The establishment of dynamic models based on game theory

3.1.1 Model objective

Establish a mathematical model to describe the dynamic speed distribution $v(s)$ of athletes during a competition and solve for the minimum time to complete the race within individual speed distributions, as well as combine the strategies of competitors to solve for the Nash equilibrium of speed distribution within the framework of game theory.

3.1.2 Competition and athlete assumptions

In this model, the total distance of the competition is denoted as L, and there are Nathletes, numbered as $i = 1,2,\dots,N$.Each athlete's stamina and speed have a dynamic relationship: remaining stamina $E_i(s)$decreases with the distance s consumed; speed $v_i(s)$is dynamically adjusted with the distance s; the power consumption of stamina is proportional to the square of the speed [6].

$$P_i(s) = k_i v_i^2(s) \tag{17}$$

In Equation (17), k_i is the physical efficiency coefficient.

3.1.3 Constraints

$$\frac{dE_i(s)}{ds} = -k_i v_i^2(s) \tag{18}$$

$$E_i(0) = E_{i,0}, E_i(L) \geq 0 \tag{19}$$

$$\mathrm{v_i(s) \leq \sqrt{\frac{E_i(s)}{k_i}}} \tag{20}$$

In the context of multiple formulas provided, "s" ranges from zero to L, representing distance, and $S_i(s)$represents the position of athlete i,satisfies $S_i'(s) = v_i(s)$.

3.1.4 Tactical interaction

Athletes' choice of speed is also influenced by their opponents' positions and speeds, which can be specifically modeled as the Catch-up Strategy, the Control Strategy, and the Personal Best Strategy [7].

The athlete's decision-making is a dynamic non-cooperative game. Each athlete i aims to minimize the completion time, and adjusts speed based on the opponent's strategy. The objective function combines race time and tactical influence [8].

$$\mathrm{J_i = \int_0^L \frac{1}{v_{i(s)}} ds - \alpha \int_0^L \sum_{j \neq i} \left(S_i(s) - S_j(s)\right) ds - \beta \int_0^L \sum_{j \neq i} (v_i(s) - v_j(s))^2 ds} \tag{21}$$

Of which, the first item: completion time of the race; the second item: reward/punishment for the gap in position, where $\alpha > 0$represents the tactical weight for catching up with opponents; the third item: penalty for speed matching, where $\beta > 0$ represents the strategy of matching the opponent's speed [9].

3.2. The solution of dynamic models based on game theory

We use the Hamilton-Pontryagin minimum principle from optimal control theory to solve for the speed distribution of athlete i whose.

3.2.1 Hamiltonian is as follows

$$\mathrm{H_i = \frac{1}{v_{i(s)}} + \lambda_i(s)\left(-k_i v_i^2(s)\right) - \alpha \sum_{j \neq i} \left(S_i(s) - S_j(s)\right) - \beta \sum_{j \neq i} (v_i(s) - v_j(s))^2} \tag{22}$$

In equation (23), $\lambda_i(s)$ is the adjoint variable, representing the "shadow price" of physical constraints. The objective is to minimize H_i.

3.2.2 Consider the optimality conditions

Take the derivative of $v_i(s)$ and set the partial derivative to zero.

$$\mathrm{\frac{\partial H_i}{\partial v_i} = -\frac{1}{v_i^2(s)} - 2\lambda_i(s) k_i v_i(s) - 2\beta \sum_{j \neq i} \left(v_i(s) - v_j(s)\right) = 0} \tag{23}$$

3.2.3 The expression for the optimal velocity $\mathbf{v_i(s)}$ is derived

$$\mathrm{v_i(s) = \frac{1}{\sqrt{1/_{v_i^4(s)} + 2\lambda_i(s) k_i}} + \beta \sum_{j \neq i} v_j(s)} \tag{24}$$

Dynamic equation for the associated variable $\lambda_i(s)$

$$\mathrm{\frac{d\lambda_i(s)}{ds} = -\frac{\partial H_i}{\partial E_i}} \tag{25}$$

Physical Dynamics Equation

$$\mathrm{\frac{dE_i(s)}{ds} = -k_i v_i^2(s)} \tag{26}$$

Among which, the initial conditions:$E_i(0) = E_{i,0}, S_i(0) = 0.$ $End\ condition$: $E_i(L) \geq 0.$

3.2.4 The definition of Nash equilibrium

When the game between athletes reaches Nash equilibrium, the speed distribution of each athlete $v_i^*(s)$ satisfies [10].

$$J_i(v_i^*(s), v_{-i}^*(s)) \le J_i(v_i(s), v_{-i}^*(s)), \forall v_i(s) \in \left[0, \sqrt{\frac{E_i(s)}{k_i}}\right] \tag{27}$$

Where, $v_i^*(s)$ is the optimal strategy of all other players except athlete i.

3.2.5 Result analysis

According to the model, using Python for auxiliary calculations, the following results were obtained. The speed distribution over 50 meters freestyle under three different strategies is shown in Figure 2.

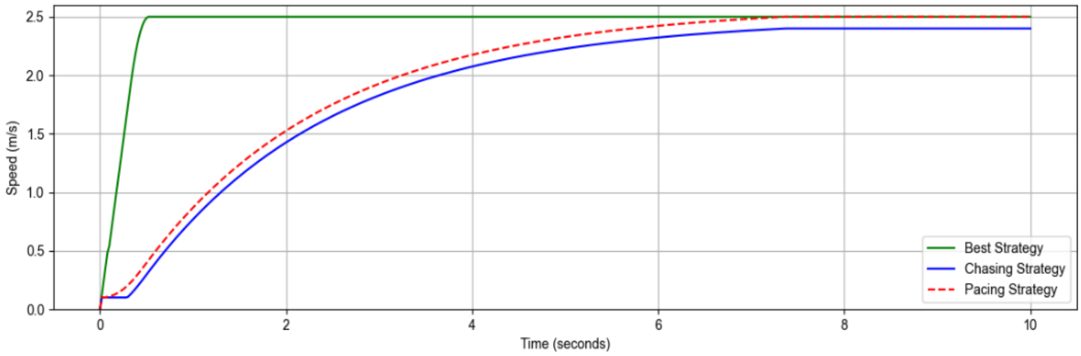

Figure 2. Speed distribution over 50 meters freestyle under three different strategies

Figure 2 shows the strategy for a 50-meter freestyle race. Athletes should start with a full sprint to gain an advantage and perform at their best. The 50-meter race is short, and compared to middle and long-distance events, there is no time for complex tactics, so the need for strategy is not strong.

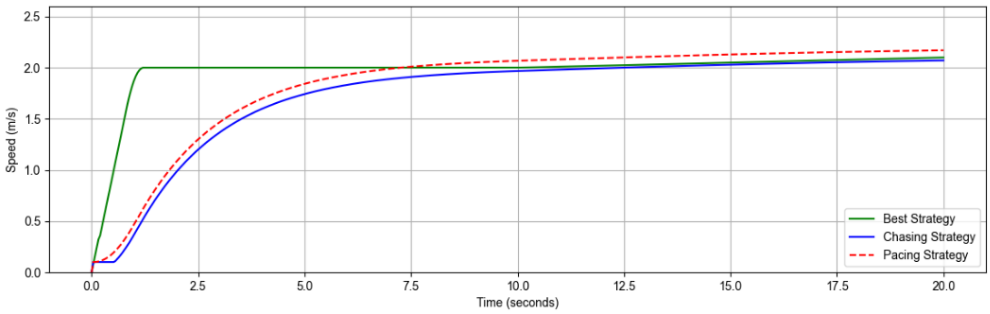

Figure 3. Speed distribution over 100 meters freestyle under three different strategies

From observing figure 3, it is evident that an effective strategy for the 100-meter freestyle is required. At the start of the race, athletes should sprint vigorously to perform at their personal best, quickly reaching their maximum speed to gain an early advantage. After the initial sprint, as energy is expended, the race enters a critical middle phase where a transition to a pacing strategy is necessary until the finish. If the pacing strategy cannot be executed, in the final stages, it is essential to abandon complex tactics and fully commit to maintaining the original best speed until the end.

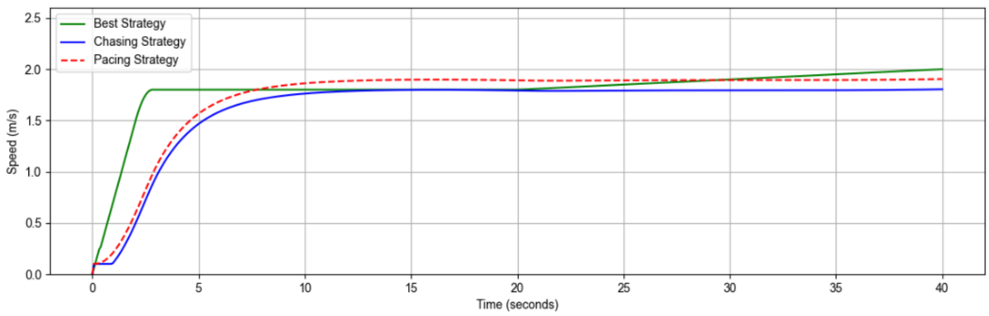

Figure 4. Speed distribution over different times for the 200-meter freestyle under three strategies

Figure 4 indicates that in the 200-meter freestyle competition, one should start by reaching their best speed quickly and sprinting to maximum velocity. After some energy is expended, a controlling strategy should be adopted in the middle stages.

In the final 200-meter sprint phase, one should give their best speed and sprint.The above method balances the tactical interaction between the athlete and their personal optimal swimming strategy, allowing the athlete to perform at their best.

4. Conclusions

This study focuses on the speed optimization problem of freestyle swimmers and establishes three models: the speed allocation model, the game-theory-based dynamic model, and the dynamic model integrating exercise physiology and optimal control theory. These models explore how to achieve optimal race strategies from three perspectives: individual speed distribution, competitive interactions, and personalized physiological characteristics.Together, these three models form a comprehensive framework from theory to practice, offering systematic solutions to improve freestyle race performance. Additionally, they demonstrate the broad potential of mathematical modeling in competitive sports.

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