Research on the Computation and Properties of the Hyperbolic Complex Determinant

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Abstract. In this paper, the calculation results and properties of determinants under hyperbolic complex numbers are studied. The hyperbolic complex number is a commutative ring consisting of two real numbers with zero divisors. This paper overcomes the difficulty that the hyperbolic complex number contains zero divisors, and calculates the decomposition of the determinant of the hyperbolic complex number by its decomposition property. This paper investigates the decomposition operation of hyperbolic complex determinants and proves that any hyperbolic complex determinant can be decomposed into two real determinants through zero factors. Based on this decomposition, the necessary and sufficient conditions for the invertibility of hyperbolic complex determinants are further derived. In addition, the property that the determinant of the product of hyperbolic complex matrices is equal to the product of their respective determinants is discussed. This will lay a foundation for the study of hyperbolic complex numbers in algebraic theory and provide a new direction for their application in physics.

Keywords: Hyperbolic Complex, Determinant, Zero Divisor Factorization.

1. Introduction

The distributed negative-positive double hyperbolic cosine algorithm has been found to perform exceptionally well in frequency estimation under various complex operating conditions in the power system. Additionally, a feed line model based on the characteristics of complex hyperbolic functions is an important requirement for accurately describing broadband harmonic resonance phenomena in distributed power plants [1-2]. In the method of studying series sum, we can investigate the properties of number theory through the utilization of hyperbolic functions and presents a closed-form expression for the summation of functions involving hyperbolic functions [3]. The utilization of hyperbolic space in graph representation learning proves advantageous for capturing the latent hierarchy when modeling bipartite graphs and has its effectiveness and high accuracy [4]. Above, the theory of hyperbolic complex numbers is studied in general, and the related work of power, series and graph representation are carried out respectively. The frequency estimation of power system, the characteristics of number theory and the modeling of bipartite graph are improved. In this paper, the algebraic properties of hyperbolic complex numbers are further discussed on the basis of the above studies.

Some scholars have studied an algorithm for approximating the logarithm of a symmetric positive definite matrix. The additive error bound of this algorithm is applicable to any approximate symmetric positive definite matrix [5]. In order to approximate the trace of the matrix, we can calculate and analyze the tail boundary by using stochastic trace estimation [6]. The stability of linear dynamic systems with quaternion coefficients can be described based on the determinant of Dieudonne constant. A matrix formed from a singular form of Woodbury matrix identity, which has its pseudo-determinant and inverse determinant, and is capable of Gaussian process regression applications [7-

8]. The above research and application of determinants have improved the approximation algorithm of determinants, quaternion coefficients and the characteristics of special matrices.

The functional determinant approach can address the Fermi polaron problem. It is an exact method for computing certain ideal quantum observables and can also obtain the spectral function of the system [9]. Some scholars have investigated that the hyperbolic neural network possesses hyperbolic reducibility, inherent singularity, and all its reducibility [10-11]. Hypformer is a novel hyperbolic transformer mainly relying on the hyperbolic geometric Lorentz model, featuring scalability of solutions and effective modules [12]. Based on the concept of hyperbolic derivatives, the generalized Cantor set on the hyperbolic number plane can be derived, and an iterative function system can be constructed through affine transformations on the hyperbolic number plane [13]. Utilizing optical hyperbolicity and tunable mechanical properties, a layered material with properties that can adapt to harsh environments can be explored [14]. Matrices and determinants are fundamental research objects in mathematics and are crucial in physics, economics and other fields. Based on the above research, this paper will further discuss the properties of matrix and determinant in the hyperbolic plane, which will provide a theoretical basis for the further application of matrix.

2. Hyperbolic complex numbers

2.1. The definition of Hyperbolic complex numbers

Hyperbolic complex numbers form an algebraic structure defined as:

$$D=\{\delta=x+ky : x, y\in R\}, \tag{1}$$

Where the unit element k satisfies the condition $k^2=1$, but it is distinct from ± 1. These numbers extend the traditional complex numbers, introducing novel properties due to the nonstandard nature of k. The following sections will outline the algebraic operations within this system and explore the implications of its unique elements.

2.2. Algebraic Structure

The addition of two hyperbolic complex numbers $\delta_1=\left(x_1+ky_1\right)$ and $\delta_2=\left(x_2+ky_2\right)$ follows standard commutative rules:

$$\begin{aligned}\delta_1+\delta_2&=\left(x_1+ky_1\right)+\left(x_2+ky_2\right)\\&=\left(x_1+x_2\right)+\left(y_1+y_2\right)k.\end{aligned} \tag{2}$$

Multiplication between hyperbolic complex numbers introduces interactions between real and imaginary components:

$$\begin{aligned}\delta_1\delta_2&=\left(x_1+ky_1\right)\left(x_2+ky_2\right)\\&=\left(x_1x_2+y_1y_2\right)+\left(x_1y_2+x_2y_1\right)k.\end{aligned} \tag{3}$$

2.3. Zero Divisors and Basis

The hyperbolic complex number system has zero divisors. For instance,

$$e=\frac{1+k}{2},\quad e^{+}=\frac{1-k}{2}, \tag{4}$$

Serve as zero divisors and can also be seen as a basis for D. These elements satisfy:

$$e^2=e,\quad \left(e^{+}\right)^2=e^{+},\quad ee^{+}=0, \tag{5}$$

Creating an idempotent structure for the ring. Thus, any hyperbolic complex number $\delta = x + ky \in D$ can be expressed in terms of e and e^{+}:

$$\delta = (x+y)e + (x-y)e^{+} = ae + be^{+}. \tag{6}$$

This decomposition aids in simplifying operations within the ring.

2.4. Positivity and Negativity

In the ring D, the condition $k^2 = 1$ allows for the definition of positive and negative hyperbolic complex numbers. Utilizing the idempotent basis, any hyperbolic complex number δ can be rewritten as:

$$\delta = ae + be^{+}. \tag{7}$$

Where $a = x + y, b = x - y$. The sign of δ is determined as follows:

(1) δ is positive if $a \succ 0$ and $b \succ 0$

(2) δ is negative if $a \prec 0$ and $b \prec 0$

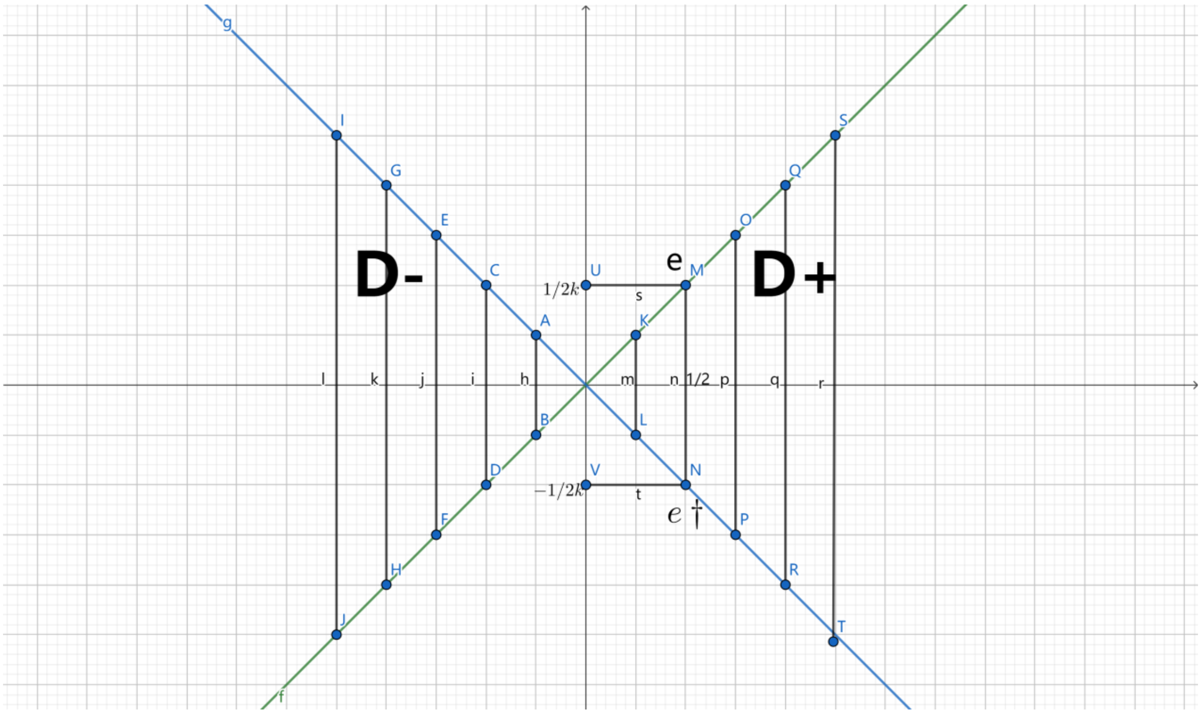

Figure 1. The Positive and Negative Hyperbolic Numbers

The Figure1 is a picture of the positive and negative hyperbolic numbers. The idempotent representation of hyperbolic numbers involves idempotents e and e^{+}. Hyperbolic numbers can be represented using idempotent "coordinates" β_1 and β_2 The set D^{+} of non-negative hyperbolic numbers can be described as:

$$\{x + ky \,|\, x \geq 0; |y| \leq x\} \tag{8}$$

Or as:

$$\{ve + \mu e^{+} \,|\, v, \mu \geq 0\} \tag{9}$$

Positive hyperbolic numbers have both idempotent components non-negative. In a geometric representation, positive hyperbolic numbers are in a quarter plane denoted by D^{+}. Negative hyperbolic numbers are symmetric to this plane. Non-positive hyperbolic numbers form the set D^-. A hyperbolic number is semi-positive if one of its coefficients is positive and the other is zero.

2.5. Modulus of Hyperbolic Complex Numbers

Given a hyperbolic number $\delta = ae + be^+$,the positive hyperbolic number

$$|\delta|_k = |ae + be^+|_k = |a|e + |b|e^+, \tag{10}$$

is called the hyperbolic modulus of δ.

$|\delta|_k$ is a modulus, since it fulfills the properties, for any $\delta, \omega \in D$, if $\delta = 0$,

$$|\delta|_k = 0 \tag{11}$$

$$|\delta\omega|_k = |\delta|_k \, |\omega|_k \tag{12}$$

$$|\delta + \omega|_k \le |\delta|_k + |\omega|_k \tag{13}$$

3. Results

Definition 3.1 Let A be an $n \times n$ hyperbolic complex numbers matrix; denote its column vectors by $\tilde{\delta}_1, \tilde{\delta}_2 \cdots \tilde{\delta}_n$: $A = (\tilde{\delta}_1, \tilde{\delta}_2 \cdots, \tilde{\delta}_n)$ .Its determinant, denoted as $\det(A)$, is

$$\det(A) = D(\tilde{\delta}_1, \tilde{\delta}_2 \cdots, \tilde{\delta}_n) = \begin{vmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{n1} & \delta_{2n} & \cdots & \delta_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}e + b_{11}e^+ & a_{12}e + b_{12}e^+ & \cdots & a_{1n}e + b_{1n}e^+ \\ a_{21}e + b_{21}e^+ & a_{22}e + b_{22}e^+ & \cdots & a_{2n}e + b_{2n}e^+ \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}e + b_{n1}e^+ & a_{n2}e + b_{n2}e^+ & \cdots & a_{nn}e + b_{nn}e^+ \end{vmatrix} \tag{14}$$

Where D is defined by formula, and $\delta_{ij} = a_{ij}e + b_{ij}e^+$.

Theorem3.1 For all $n \times n$ Hyperbolic complex numbers matrices A,

$$\begin{aligned} \det(A) &= \begin{vmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{n1} & \delta_{2n} & \cdots & \delta_{nn} \end{vmatrix} \\ &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{2n} & \cdots & a_{nn} \end{vmatrix} e + \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{2n} & \cdots & b_{nn} \end{vmatrix} e^+ \end{aligned} \tag{15}$$

Proof.

$$\det(A)=\begin{vmatrix}\delta_{11} & \delta_{12} & \cdots & \delta_{1n}\\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ \delta_{n1} & \delta_{2n} & \cdots & \delta_{nn}\end{vmatrix}=\begin{vmatrix}a_{11}e+b_{11}e^{+} & a_{12}e+b_{12}e^{+} & \cdots & a_{1n}e+b_{1n}e^{+}\\ a_{21}e+b_{21}e^{+} & a_{22}e+b_{22}e^{+} & \cdots & a_{2n}e+b_{2n}e^{+}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n1}e+b_{n1}e^{+} & a_{n2}e+b_{n2}e^{+} & \cdots & a_{nn}e+b_{nn}e^{+}\end{vmatrix}$$

$$=\sum_{j_1,j_2\cdots j_n}(-1)^{\tau(j_1,j_2\cdots j_n)}\delta_{1j_1}\delta_{2j_2}\cdots\delta_{nj_n} \tag{16}$$

$$=\sum_{j_1,j_2\cdots j_n}(-1)^{\tau(j_1,j_2\cdots j_n)}(a_{1j_1}e+b_{1j_1}e^{+})(a_{2j_2}e+b_{2j_2}e^{+})\cdots(a_{nj_n}e+b_{nj_n}e^{+}).$$

According to equation (5), we have

$$\det(A)=\sum_{j_1,j_2\cdots j_n}(-1)^{\tau(j_1,j_2\cdots j_n)}(a_{1j_1}a_{2j_2}\cdots a_{nj_n}e+b_{1j_1}b_{2j_2}\cdots b_{nj_n}e^{+})$$

$$=\sum_{j_1,j_2\cdots j_n}(-1)^{\tau(j_1,j_2\cdots j_n)}(a_{1j_1}a_{2j_2}\cdots a_{nj_n})e+\sum_{j_1,j_2\cdots j_n}(-1)^{\tau(j_1,j_2\cdots j_n)}(b_{1j_1}b_{2j_2}\cdots b_{nj_n})e^{+} \tag{17}$$

$$=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n1} & a_{2n} & \cdots & a_{nn}\end{vmatrix}e+\begin{vmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ b_{n1} & b_{2n} & \cdots & b_{nn}\end{vmatrix}e^{+}.$$

Theorem3.2 The necessary and sufficient condition for matrix (A) invertibility is:
$\det(A_1)\neq 0$ and $\det(A_2)\neq 0$

Where $A_1=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n1} & a_{2n} & \cdots & a_{nn}\end{vmatrix}$ and $A_2=\begin{vmatrix}b_{11} & b_{12} & \cdots & b_{1n}\\ b_{21} & b_{22} & \cdots & b_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ b_{n1} & b_{2n} & \cdots & b_{nn}\end{vmatrix}$.

Proof. According to the matrix (A), we can divide matrix (A) into: $|A|=|A_1|e+|A_2|e^{+}$

First, we prove the necessity.

If matrix (A) is invertible, then there exists a matrix such that

$$AB=BA=I. \tag{18}$$

Let

$$A=A_1e+A_2e^{+},\quad B=B_1e+B_2e^{+}, \tag{19}$$

So

$$AB=A_1B_1e+A_2B_2e^{+}=I=e+e^{+}, \tag{20}$$

$$A_1B_1=I \text{ and } A_2B_2=I. \tag{21}$$

Since

$$\det(A_1)\neq 0 \text{ and } \det(A_2)\neq 0. \tag{22}$$

Therefore, A_1 and A_2 are invertible real matrices. So

$$B_1=A_1^{-1},\quad B_2=A_2^{-1}. \tag{23}$$

Next, we prove the sufficiency.

If $\det(A_1) \neq 0$ and $\det(A_2) \neq 0$ .We have A_1 and A_2 are invertible real matrices. Construct a matrix,

$$B = A_1^{-1}e + A_2^{-1}e^{+}, \tag{24}$$

$$\begin{aligned} AB &= (A_1 e + A_2 e^{+})(A_1^{-1}e + A_2^{-1}e^{+}) \\ &= (A_1 A_1^{-1})e + (A_2 A_2^{-1})e^{+} \\ &= e + e^{+} \\ &= I \end{aligned} \tag{25}$$

$$\begin{aligned} BA &= (A_1^{-1}e + A_2^{-1}e^{+})(A_1 e + A_2 e^{+}) \\ &= (A_1^{-1}A_1)e + (A_2^{-1}A_2)e^{+} \\ &= e + e^{+} \\ &= I. \end{aligned} \tag{26}$$

Therefore, the conclusion under the above conditions holds.
Theorem3.3 For all pairs of $n \times n$ hyperbolic complex numbers matrices A and B,

$$\det(AB) = \det(A)\det(B). \tag{27}$$

Proof: According to equation (7), the *ij*th column of δ is δ_{ij} .

$$\delta_{ij} = a_{ij}e + b_{ij}e^{+}, \tag{28}$$

So

$$A = A_1 e + A_2 e^{+}, \tag{29}$$

Where $A_1 = (a_{ij})_{n\times n}$ and $A_2 = (b_{ij})_{n\times n}$ .According to equation (5),

$$\begin{aligned} AB &= (A_1 e + A_2 e^{+})\left(B_1 e + B_2 e^{+}\right) \\ &= A_1 B_1 e + A_2 B_2 e^{+}. \end{aligned} \tag{30}$$

According to equation Theorem3.1,

$$\begin{aligned} \det(A) &= \det(A_1 e + A_2 e^{+}) \\ &= \det(A_1 e) + \det(A_2 e^{+}) \\ &= \det(A_1)e + \det(A_2)e^{+}. \end{aligned} \tag{31}$$

Let

$$A = A_1 e + A_2 e^{+}, \quad B = B_1 e + B_2 e^{+} \tag{32}$$

$$\begin{aligned} AB &= (A_1 e + A_2 e^{+})\left(B_1 e + B_2 e^{+}\right) \\ &= A_1 B_1 e + A_2 B_2 e^{+}, \end{aligned} \tag{33}$$

The left-hand side expression:

$$\begin{aligned} \det(AB) &= \det(A_1 B_1)e + \det(A_2 B_2)e^{+} \\ &= \det(A_1)\det(B_1)e + \det(A_2)\det(B_2)e^{+}, \end{aligned} \tag{34}$$

The right-hand side expression:

$$\begin{aligned}&\det(A)\det(B)=[\det(A_1)e+\det(A_2)e^{+}][\det(B_1)e+\det(B_2)e^{+}]\\&=\det(A_1)\det(B_1)e+\det(A_2)\det(B_2)e^{+},\end{aligned}\tag{35}$$

So

$$\begin{aligned}&\det(AB)=\det(A_1B_1)e+\det(A_2B_2)e^{+}\\&=\det(A_1)\det(B_1)e+\det(A_2)\det(B_2)e^{+}\\&=\det(A)\det(B).\end{aligned}\tag{36}$$

4. Conclusion

The trend of mass data in power system provides a basis for load characteristic analysis and prediction model establishment, but the classical load forecasting method cannot afford such a huge time and computing resource consumption. The problem of over fitting in large sample set will affect the prediction accuracy. In this paper, a power load forecasting model is built by using the BP neural network model, making full use of the powerful data processing function of Clementine and preventing the over fitting function. The experimental results show that the BP neural network model has good predictability and robustness, and has a certain practical application value.

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