The Integral and Limit Exchange Theorem for Hyperbolic Complex Columns

Ruiying Guo[1, *, #], Meiyi Shan[2, #], Renjie Tong[3, #]

[1]College of Mathematics and System Sciences, Xinjiang University, Urumqi, China, 830049

[2]School of Science, Tianjin University of Technology and Education, Tianjin, China, 300222

[3]School of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing, China, 402247

*Corresponding author: grygry0128@163.com

#These authors contributed equally.

Abstract. This paper delves into hyperbolic numbers and their generalizations like bicomplex and hyperbolic complex numbers. By analyzing prior studies, it explores their mathematical properties, including basic concepts, function sequences, and integral commutativity.The key finding is the proof of the theorem on the interchange of integral and limit of hyperbolic variable function numbers, which is significant for theoretical research.These numbers and functions have wide ranging applications in physics. For example, they are used in visible frequency hyperbolic plasmon polaritons and hyperbolic metamaterial based photonic weyl nodal line semimetals, providing new ideas for physics research.

Keywords: Hyperbolic Complex Numbers, $Cl_{0,1}$-Antidifferentiable, $|h|_{\mathrm{H}}$.

1. Introduction

In the vast field of mathematics, complex numbers and their generalized forms have always been the focus of research. In recent years, numerous scholars have been committed to exploring different dimensions and structural characteristics of complex numbers, aiming to uncover deeper mathematical mysteries[1-4]. Richter's research work has blazed a new trail in this field. Based on the concepts of vector products and powers, he conducted in-depth research on hyperbolic complex algebraic structures in two-dimensional, three-dimensional, and four-dimensional spaces[5]. Through the improvement of arrow multiplication and the geometric interpretation of random vector products, Richter successfully extended the systematic method to the field of hyperbolic complex numbers and explored the properties of quadratic vector equations and hyperbolic holomorphic functions, further enriching the content of complex number theory.

On this basis, Halici further introduced bicomplex numbers with coefficients from the complex Fibonacci sequence. This innovation not only expands the application range of bicomplex numbers but also provides them with a richer mathematical structure. By examining the dual form of the newly defined numbers, Halici gave fundamental identities such as Cassini and Catalan, making important contributions to the development of bicomplex numbers[6].

Meanwhile, Gargoubi's research revealed that the hyperbolic number system is the only generalization of real numbers to two-dimensional Archimedes f-algebras and established various properties of hyperbolic numbers related to the f-algebraic structure[7]. This discovery not only deepens our understanding of hyperbolic numbers but also provides new interpretations and perspectives for two-dimensional space-time geometry.Akar emphasis is placed on the introduction of arithmetic operations on pairs of hyperbolic numbers and hyperbolic complex numbers constituting commutative rings. Then by studying the dual hyperbolic and hyperbolic complex numerical functions, we find that these functions have similar properties[8].

Meanwhile, hyperbolic functions also have extensive applications in the field of physics[9-12]. This paper aims to continue this research trend and explore the latest progress of hyperbolic numbers and their generalized forms (such as bicomplex numbers, hyperbolic complex numbers, etc.) in mathematical theory and physical applications. By utilizing the zero factorization property of hyperbolic numbers and decomposing the hyperbolic function sequence into two real function sequences, we obtain the commutative theorem for hyperbolic integrals and limits. By summarizing and analyzing the research achievements of predecessors, this paper will further excavate the internal characteristics and potential applications of these structures, providing new ideas and inspirations for the development of complex number theory and related fields.

2. The basic of hyperbolic numbers

The set with the following formula is a hyperbolic numbers:

$$\mathrm{H} = \left\{ h = x + iy \middle| x, y \in \mathbb{R}, i^2 = 1, h \neq \pm 1 \right\}, \tag{1}$$

which has a significant difference from complex numbers:

$$\mathbb{C} = \left\{ c = x + jy \middle| x, y \in \mathbb{R}, j^2 = -1 \right\}, \tag{2}$$

many literature refer to these numbers as double numbers, spacetime numbers, perplex numbers or split complex numbers.

We define the real parts and imaginary parts of hyperbolic numbers as $\mathrm{Im}(h) = y$ and $\mathrm{Im}(h) = y$, and define the conjugate of as $\bar{h} = x - iy$.

The ring of hyperbolic number has idempotent elements zero-divisors $0 = ee^{\dagger}$, and the idempotent base $\left\{e, e^{\dagger}\right\} \in \mathrm{H}$, where $e = \frac{1+i}{2}$ and $e^{\dagger} = \frac{1-i}{2}$, exists operations as:

$$e^{\dagger^2} = e^{\dagger}, e^2 = e, \ e + e^{\dagger} = 1, e - e^{\dagger} = i. \tag{3}$$

There exists an equation

$$h = x + iy = (x + y)e + (x - y)e^{\dagger} = \alpha e + \beta e^{\dagger}, h \in \mathrm{H}. \tag{4}$$

Let $h_1 = (x_1 + iy_1) = \alpha_1 e + \beta_1 e^{\dagger}, h_2 = (x_2 + iy_2) = \alpha_2 e + \beta_2 e^{\dagger}$, and the Addition and multiplication of set H are given as follows:

$$\begin{aligned} h_1 + h_2 &= \left(\alpha_1 e + \beta_1 e^{\dagger}\right) + \left(\alpha_2 e + \beta_2 e^{\dagger}\right) \\ &= \left(\alpha_1 + \alpha_2\right)e + \left(\beta_1 + \beta_2\right)e^{\dagger}, \end{aligned} \tag{5}$$

$$\begin{aligned} h_1 h_2 &= \left(\alpha_1 e + \beta_1 e^{\dagger}\right)\left(\alpha_2 e + \beta_2 e^{\dagger}\right) \\ &= \left(\alpha_1 \alpha_2\right)e + \left(\beta_1 \beta_2\right)e^{\dagger}. \end{aligned} \tag{6}$$

The set of hyperbolic numbers can be divided into H^+ and H^- based on the sign of α and β, called the set of non-negative and non-positive hyperbolic numbers , written separately as:

$$\mathrm{H}^+ = \left\{\alpha e + \beta e^{\dagger}, \alpha \geq 0, \beta \geq 0\right\}, \tag{7}$$

$$\mathrm{H}^- = \left\{\alpha e + \beta e^{\dagger}, \alpha \leq 0, \beta \leq 0\right\}. \tag{8}$$

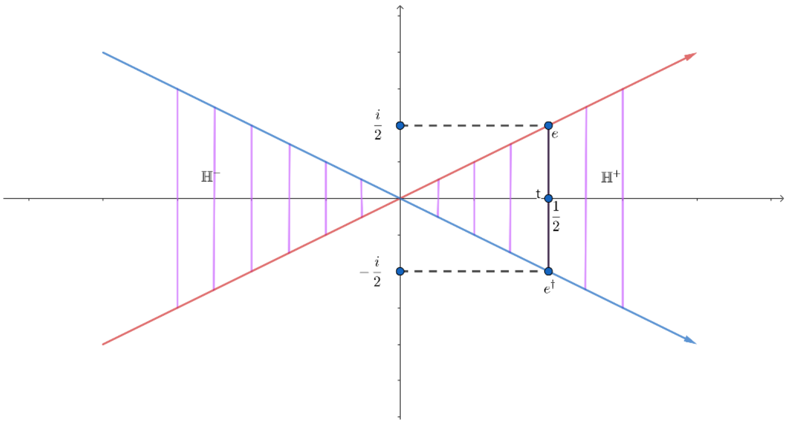

Figure1: the positive and negative hyperbolic numbers

Through Figure1, we can see that the point (x, y) corresponds to the hyperbolic number $h = x + iy$. Geometrically, positive hyperbolic numbers lie in the quarter plane denoted by H^{+} . The quarter plane symmetric with respect to the origin corresponds to negative hyperbolic numbers.

The partial order of H using the symbol $\circ$, ω is H-greater-or H-less- than h can be represented separately as:

$$h \preceq \omega \text{ iff } h - \omega \in \mathbb{H}^{+},\ h,\ \omega \in \mathbb{H}, \tag{9}$$

$$\omega \preceq h \text{ iff } h - \omega \in \mathbb{H}^{-},\ h,\ \omega \in \mathbb{H}. \tag{10}$$

Like this, we can express,

$$0 \preceq h \text{ iff } h \in \mathbb{H}^{+},\ h \preceq 0 \text{ iff } h \in \mathbb{H}^{-}, \tag{11}$$

$$0 \prec h \text{ iff } h \in H^{+} - \{0\},\ h \prec 0 \text{ iff } h \in H^{+} - \{0\}, \tag{12}$$

and for h_1 and h_2 , we have $h_1 \circ h_2$ iff $\alpha_1 \le \beta_1$ and $\alpha_2 \le \beta_2$.

When $h_n = \alpha_n e + \beta_n e^{\dagger},\ n \in \{1, 2\},\ h = \varepsilon_1 e + \varepsilon_2 e^{\dagger}$ the close hyperbolic interval

$$\begin{aligned} [h_1, h_2]_{\mathbb{H}} &= \{h \in \mathbb{H} \mid h_1 \preceq h \preceq h_2\} \\ &= \{\alpha_1 \le \varepsilon_1 \le \alpha_2 \text{ and } \beta_1 \le \varepsilon_2 \le \beta_2\}. \end{aligned} \tag{13}$$

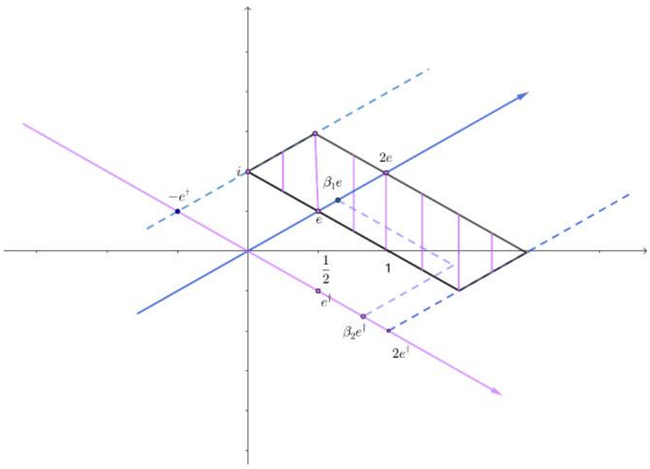

Figure 2: the hyperbolic interval $[i, 2]_{H}$

Through Figure 2, we can see that h_1 and h_2 be two subsets of H . The set h_1 represents the set of all elements in h_1 multiplied by -1. The sum of all $\omega + n$ where ω belongs to h_1 and n belongs to h_2 is represented as $h_1 + h_2$. Sets $h_1 - h_2$ are defined in the same way.

The modulus of hyperbolic numbers is a positive real number as :

$$|h|_{\mathrm{H}} = |\alpha e + \beta e^{\dagger}| = |\alpha| e + |\beta| e^{\dagger}. \tag{14}$$

Definition1 A sequence $\{h_n\}_{n\in\mathbb{N}}$ of hyperbolic number H-converges to the hyperbolic number h_0 , if for almost every strictly positive hyperbolic number ε , there exists $N \in \mathbb{N}, n \geq N$, such that $|Z_n - Z_0|_i \prec \varepsilon$, then we say that Z_0 is the limit of the sequence which we write as $\lim_{n\to\infty} Z_n = Z_0$, using the idempotent representations $|\alpha_n - \alpha_0| < \varepsilon_1$ and $|\beta_n - \beta_0| < \varepsilon_2$.

The hyperbolic integration of a hyperbolic function $\phi(\varpi) = \phi_1(x,y) + j\phi_2(x,y)$ is evaluated with respect to some four-dimensional curve H in C_2 and is defined as $I = \int_H \phi(\omega) \otimes d\omega$, $d\omega = (dx, dy)$.

Lemma2. If $f_1 : [\alpha_1, \beta_1] \to \mathbb{R}$ and $f_2 : [\alpha_2, \beta_2] \to \mathbb{R}$ are holomorphic functions in $\mathbb{R}$ on the domains $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ respectively. Then we can define a function $f : [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \to \mathbb{R}_2$ as

$$f(\omega) = f_1(\alpha) e + f_2(\beta) e^{\dagger} \tag{15}$$

where $\omega = \omega_1 e_1 + \omega_2 e_2 \Rightarrow d\omega = (d\omega_1) e_1 + (d\omega_2) e_2,\ \forall \omega \in [\alpha_1, \beta_1] \times_e [\alpha_2, \beta_2] \subseteq \mathbb{R}_2$.

Now we have

$$\int_H f(\omega) \otimes d\omega = \left\{ \int_{\gamma_1} f_1(\omega_1) \otimes d\omega_1 \right\} e_1 + \left\{ \int_{\gamma_2} f_2(\omega_2) \otimes d\omega_2 \right\} e_2 \tag{16}$$

where $H : \omega = \omega(t),\ \omega(t) = \omega_1(t) e_1 + \omega_2(t) e_2$ for $r \le t \le s$.

Consider the differential operator $\overline{\nabla} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, let f be a split hyperbolic valued function of a split hyperbolic variable , then $\overline{\nabla} f(h_0) = 0 \Leftrightarrow f$ is $C\ell_{0,1}$-differentiable at h_0 . The split complex valued function f is $C\ell_{0,1}$-differentiable iff $f(h) = f_1(\alpha) e + f_2(\beta) e^{\dagger}$, also, f is $C\ell_{0,1}$-antidifferentiable iff $f = g_1(\beta) e + g_2(\alpha) e^{\dagger}$.

3. Results

Theorem3.1 (The Theorem on the Interchange of Integral and Limit of Hyperbolic Variable Function Numbers) Let $\{f_n\}$ be a sequence of functions on the set H , if for almost every $\varepsilon = \varepsilon_1 e + \varepsilon_1 e^{\dagger} \succ 0$ and there exists $N > 0$,almost every $n > N$, $h = \alpha e + \beta e^{\dagger}$ on the close hyperbolic interval $[h_1, h_2]_{\mathrm{H}}$ such that $|f_n(h) - f(h)|_{\mathrm{H}} \prec \varepsilon$ and f is $C\ell_{0,1}$-differentiable, then

$$\int_{[h_1,h_2]_{\mathrm{H}}} \lim_{n\to\infty} f(h) \otimes dh = \lim_{n\to\infty} \int_{[h_1,h_2]_{\mathrm{H}}} f_n(h) \otimes dh. \tag{17}$$

Proof :By the conditions, we have

$$\begin{aligned} \left|f_n(h)-f(h)\right|_{\mathrm{H}} &= \left|f_{1,n}(\alpha)e+f_{2,n}(\beta)e^{\dagger}-f_1(\alpha)e+f_2(\beta)e^{\dagger}\right|_{\mathrm{H}} \\ &= \left|f_{1,n}(\alpha)-f_1(\alpha)\right|_{\mathrm{H}} e+\left|f_{2,n}(\beta)+f_2(\beta)\right|_{\mathrm{H}} e^{\dagger} \\ &\prec \varepsilon=\varepsilon_1+\varepsilon_2 \end{aligned} \tag{18}$$

thus

$$\begin{aligned} &\left|f_{1,n}(\alpha)-f_1(\alpha)\right|_{\mathrm{H}} \prec \varepsilon_1 \\ &\left|f_{2,n}(\beta)-f_2(\beta)\right|_{\mathrm{H}} \prec \varepsilon_2. \end{aligned} \tag{19}$$

Further, using Lemma2, we get

$$\int_{[h_1,h_2]_{\mathrm{H}}} f(h)\otimes dh=\left[\int_{\alpha_1}^{\beta_1} f_1(\alpha)d\alpha\right]e+\left[\int_{\alpha_2}^{\beta_2} f_2(\beta)d\beta\right]e^{\dagger}. \tag{20}$$

Also, by the properties of integrals, we have

$$\begin{aligned} &\left|\int_{[h_1,h_2]_{\mathrm{H}}} f_n(h)\otimes dh-\int_{[h_1,h_2]_{\mathrm{H}}} f(h)\otimes dh\right|_{\mathrm{H}} \\ &=\left|\int_{\alpha_1}^{\beta_1} f_{1,n}(\alpha)d\alpha e-\int_{\alpha_2}^{\beta_2} f_{2,n}(\beta)d\beta e^{\dagger}-\int_{\alpha_1}^{\beta_1} f_1(\alpha)d\alpha e-\int_{\alpha_2}^{\beta_2} f_2(\beta)d\beta e^{\dagger}\right|_{\mathrm{H}} \\ &=\left|\int_{\alpha_1}^{\beta_1}\left[f_{1,n}(\alpha)-f_1(\alpha)\right]d\alpha e+\int_{\alpha_2}^{\beta_2}\left[f_{2,n}(\beta)-f_2(\beta)\right]d\beta e^{\dagger}\right|_{\mathrm{H}} \\ &\circ\ \int_{\alpha_1}^{\beta_1}\left[f_{1,n}(\alpha)-f_1(\alpha)\right]d\alpha e+\int_{\alpha_2}^{\beta_2}\left[f_{2,n}(\beta)-f_2(\beta)\right]d\beta e^{\dagger} \\ &\circ\ \varepsilon_1(\beta_1-\alpha_1)e+\varepsilon_2(\beta_2-\alpha_2)e^{\dagger} \\ &=\varepsilon(h_1-h_2) \end{aligned} \tag{21}$$

Then

$$\lim_{n\to\infty}\int_{[h_1,h_2]_{\mathrm{H}}} f_n(h)\otimes dh=\int_{[h_1,h_2]_{\mathrm{H}}} f(h)\otimes dh=\int_{[h_1,h_2]_{\mathrm{H}}}\lim_{n\to\infty} f(h)\otimes dh. \tag{22}$$

4. Conclusions

This paper discusses the commutativity of the limit of function sequences and integrals in the theory of hyperbolic complex analysis and obtains the theorem on the interchange of the limit and integral of hyperbolic complex function sequences. Based on combing through previous achievements, it deeply analyzes their mathematical properties and successfully proves the interchangeability theorem of integrals and limits of hyperbolic variable function numbers. Due to the zero - divisor decomposability of hyperbolic complex functions, the interchange theorem is obtained by approximating the decomposition of the limit function sequence. This will lay a foundation for the further research on the integral theory and limit theory of hyperbolic functions. At the same time, it will provide impetus for the applications of the hyperbolic complex analysis theory in physics.

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