Solutions of the Irrotational Euler Equation

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Abstract. The irrotational Euler's equation plays a significant role in fluid dynamics, and an in-depth study of its solutions helps to better understand the motion of fluids. This paper primarily addresses the analytical solutions of the irrotational Euler's equation in the absence of external forces. Analytical solutions for incompressible, steady-state fluids, as well as numerical solutions for cases where density is a function of time, are solved. Numerical solutions for the two-dimensional wave equation are solved and the images of the numerical solutions changing over time are plotted.

Keywords: Euler's Equation, Continuity Equation, Wave Equation.

1. Introduction

In the field of fluid mechanics, the Navier-Stokes (N-S) equations play a central role. When the viscosity is neglected, the N-S equations degenerate into the Euler equations, which describe the dynamics of an ideal fluid under the conditions of no viscosity and no external forces when the velocity field is irrotational. Originating from the fundamental principles of classical mechanics, these equations are the foundation for understanding fluid statics and dynamics. The irrotational property of the fluid, that is, the curl of the fluid velocity field being zero, simplifies the complexity of fluid motion, making the mathematical model more manageable.

The continuity equation in fluid mechanics is one of the fundamental equations describing the conservation of mass of a fluid. It plays an extremely important role in the study of fluid dynamics and its engineering applications. Based on the principle of mass conservation, the continuity equation can be stated as the mass flow rate entering a certain area being equal to the mass flow rate leaving the area within a unit of time under the condition of no sources or sinks.

The water wave equation is a class of partial differential equations used to describe the propagation and evolution of water waves. They have significant application value in coastal engineering, ocean engineering, and the study of related geophysical phenomena. In addition, the research on forward and inverse algorithms of wave equations is also of great theoretical and practical application value in fields such as geological exploration, tunnel seismic, and ultrasonic CT imaging.

Deng Ruijuan, Chen Qianqian studied a class of general solutions for high-order Euler equations. First, for the second and third order cases, by using the classic variable substitution method to transform the variable-coefficient Euler equations into constant-coefficient linear equations, the equations of the same order were obtained, and application examples were given [1]. Then, this result was extended to obtain the solution method for the n-order case. Du Dongqing, Du Xianghan et al. studied the numerical simulation of the continuity equation in phase change heat transfer in porous media [2]. Using the continuity equation in fluid mechanics, a certain range of research was conducted on heat and mass transfer of fluids in porous media with phase change. Numerical simulation and image analysis concluded that under two-phase flow conditions, the fluid and solid have thermal non-equilibrium characteristics, the two-phase region has non-isothermal characteristics, and the temperature of the fluid and solid both show an N-shaped distribution, which has an important impact on the mathematical modeling of the process. Ren Qinqin, Bai Xiaoya et al. studied a high-precision numerical scheme for solving the Euler equations [3]. To optimize the computational performance of the weighted essentially non-oscillatory scheme in fluid mechanics, based on the research and analysis of the weighted methods of the WENO-JS and WENO-M schemes, a new local smoothness factor was derived based on the generalized indivisible difference operator, and the WENO-NS scheme was constructed. The new coupled scheme resulted in high precision and good stability in the computational results. Zhao Yingjie, Wang Zhiliang, Sun Xiongwei et al. introduced the ONADM method for solving the seismic wave equation in the elastic medium for the two-dimensional wave equation, and then used this method for numerical simulation of the two-dimensional wave equation in homogeneous media and horizontal layered velocity models [4]. The results showed that in homogeneous media, clear, dispersion-free wave field snapshots were obtained, and the seismic records of the horizontal layered velocity model were clear, with direct waves and reflected waves being distinctly separated. Moreover, solutions and applications of irrotational Euler equations have been studied by many authors, see, for example [5-11].

This paper uses the method of separation of variables to solve the analytical solution of the irrotational Euler equation without external forces. Analytical solutions for incompressible, steady-state fluids, as well as numerical solutions for cases where density is a function of time, were solved. For the case where density is a function of time, we assume that the velocity field is irrotational, and the final equation is equivalent to the Poisson equation. A series of Poisson equations were numerically solved. We also solved the numerical solutions for the two-dimensional wave equation under several different external fields and plotted the images of the numerical solutions changing over time.

2. Euler's Equation

2.1. An exposition of the irrotational Euler equations.

The Euler equations under the condition of irrotationality is as follows:

$$\partial_t \phi + (\nabla \phi)^2 = g \tag{1}$$

Solving the irrotational Euler equations in the absence of external forces. The Euler equations under irrotational conditions are given by $\partial_t \phi + (\nabla \phi)^2 = g$. The process of solving these equations is as follows: If $g = 0$, then we assume:

$$\phi(x, y, z, t) = T(t)\varphi(x, y, z) \tag{2}$$

The following can be derived:

$$T'(t) \cdot \varphi(x, y, z) + \Big(\nabla\big(T(t) \cdot \varphi(x, y, z)\big)\Big)^2 = 0 \tag{3}$$

That is:

$$\frac{T'(t)}{T^2(t)} = -\frac{\big(\nabla\varphi(x,y,z)\big)^2}{\varphi(x,y,z)} = -k \tag{4}$$

Hence, it can be obtained that:

$$\frac{T'(t)}{T^2(t)} = -k \tag{5}$$

This is the result:

$$T(t) = \frac{1}{kt - C} \tag{6}$$

3. Continuity Equation

3.1. Description of the Continuity Equation For the continuity equation

For the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{7}$$

3.2. Incompressible Fluid

The continuity equation is given by:

$$\nabla \cdot \vec{v} = 0 \tag{8}$$

That is:

$$\vec{v} = \nabla \varphi \tag{9}$$

3.3. Steady Flow

For steady flow, the density ρ is known, that is, ρ is a function of space, and we have:

$$\nabla \cdot (\rho \vec{v}) = 0 \tag{10}$$

Let $\rho \vec{v} = \nabla \phi$, then we have:

$$\vec{v} = \frac{\nabla \phi}{\rho} \tag{11}$$

3.4. Density as a Function of Time Only

If ρ is a function of time $\rho = \rho(t)$, then:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \vec{v} = 0 \tag{12}$$

It can be derived that:

$$\nabla \vec{v} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t} \tag{13}$$

If in the flow field $\nabla \times \vec{v} = 0$, it is referred to as an irrotational flow. For an irrotational fluid, a potential function ϕ is defined: $\vec{v} = \nabla \varphi$. Substituting this into the equation $\nabla \vec{v} = -\frac{1}{\rho}\frac{\partial \rho}{\partial t}$ yields:

$$\nabla \cdot \nabla \varphi = \nabla^2 \varphi = -\frac{1}{\rho}\frac{\partial \rho}{\partial t} = f(t) \tag{14}$$

The equation is transformed into a Poisson equation. We use the method of separation of variables, separating φ into a function $T(t)$ that depends only on time and a function $R(r)$ that depends only on space, then we can obtain $T(t)\nabla^2 R(r) = f(t)$, that is, $\nabla^2 R(r) = \frac{f(t)}{T(t)} = \lambda$, where λ is an arbitrary constant.

The numerical solution of the Poisson equation when $f(t)=0$ is shown in Figure 1. At this time, the equation only has a trivial constant solution.

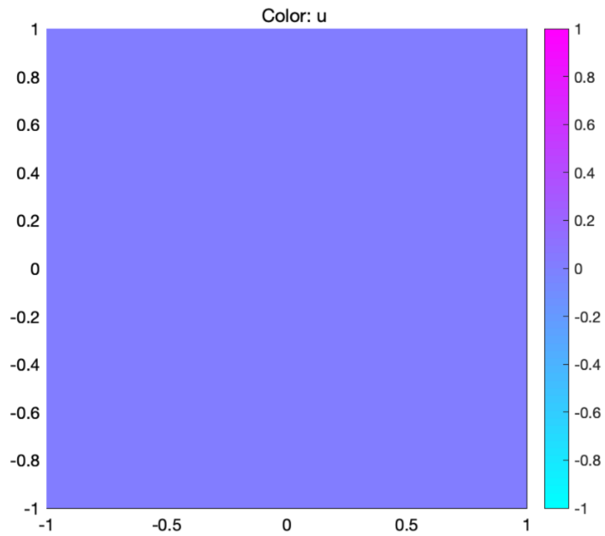

Figure 1. Poisson's equation with f(t)=0

Figure 2. shows numerical solution of Poisson's equation with *f*(*t*)=1. The numerical solution, as depicted in the figure, exhibits C4 symmetry and reaches its maximum value at 0.

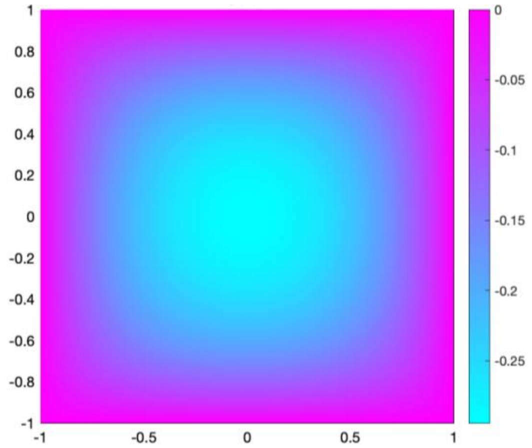

Figure 2. Poisson's equation with f(t)=1

If *f*(*t*) is a function of time, then the solution $R(r)$ to $\nabla^2 R(r) = \lambda$ (where λ is an arbitrary constant) is formally identical to the solution when $\nabla^2\varphi = f(t) = 1$. This is illustrated in Figure 3.

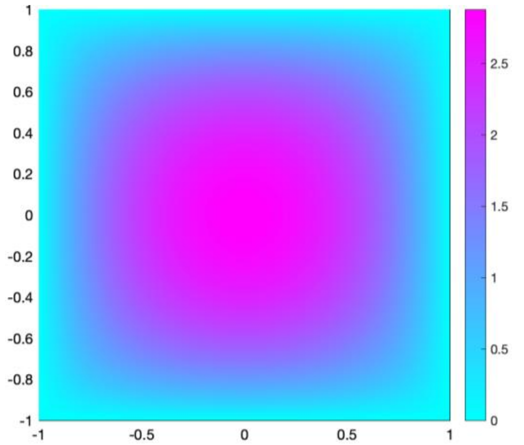

Figure 3. Poisson's equation with f(t)=t

4. Two-Dimensional Wave Equation

4.1. Description of the Two-Dimensional Wave Equation

The two-dimensional wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x, y, t) \tag{15}$$

4.2. Method of Separation of Variables

When $f(x, y, t) = 0$, the method of separation of variables is applied to solve as follows:

Assuming $u(x, y, t) = X(x)Y(y)T(t)$, substituting into the wave equation yields:

$$X(x)Y(y)T''(t) = c^2X''(x)Y(y)T(t) + c^2X(x)Y''(y)T(t)$$

Separation of variables: Dividing both sides of the above equation by $c^2X(x)Y(y)T(t)$, we obtain:

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$$

The equation on both sides is a constant: Since the left side depends only on time t and the right side depends only on the spatial variables x and y , the equation on both sides equals the constant, set as $-\lambda$, where $\lambda > 0$:

$\frac{T''(t)}{c^2T(t)}$ is solved as a constant coefficient ordinary differential equation.

$$T''(t) + \lambda c^2 T(t) = 0$$
$$X''(x) - \mu X(x) = 0$$
$$Y''(y) - \nu Y(y) = 0$$

Here, μ and ν are constants that are determined by λ.

4.3. Numerical Solutions

When $f(x, y, t) = 0$, numerical solutions of the wave equation can be obtained and visualized. The graphical representation of the wave equation's solutions is presented as follows:

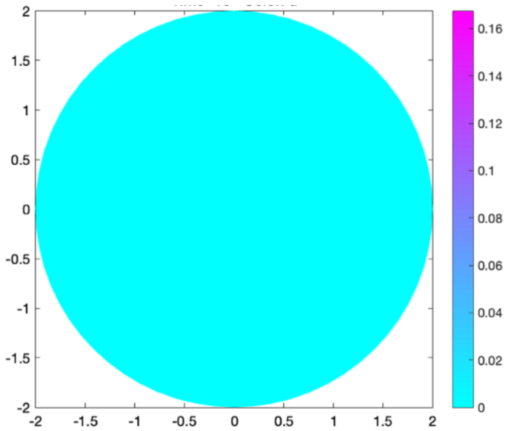

Figure 4. Poisson's equation with $f(x, y, t) = 0$

When $f(x, y, t) = 1$, the numerical solution of the wave equation is obtained and is depicted in Figure 5.

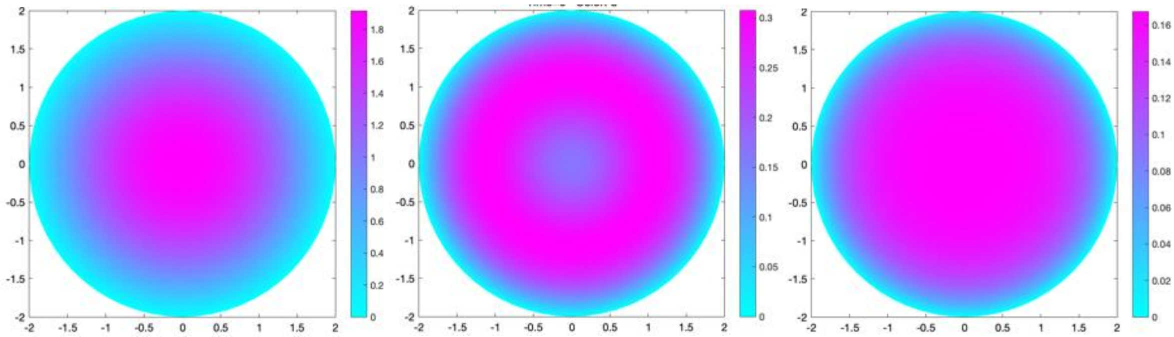

Figure 5. Poisson's equation with $f(x, y, t) = 1$

When $f(x, y, t)$ is solely dependent on time t, for instance, when $f(x, y, t) = t$, the numerical solution obtained is as follows:

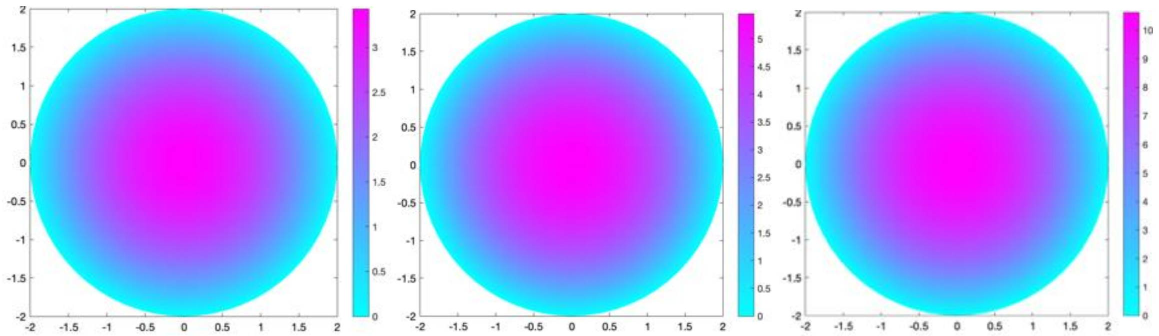

Figure 6. Poisson's equation with $f(x, y, t) = t$

When $f(x, y, t) = e^{-t^2}$, the numerical solution obtained is as follows:

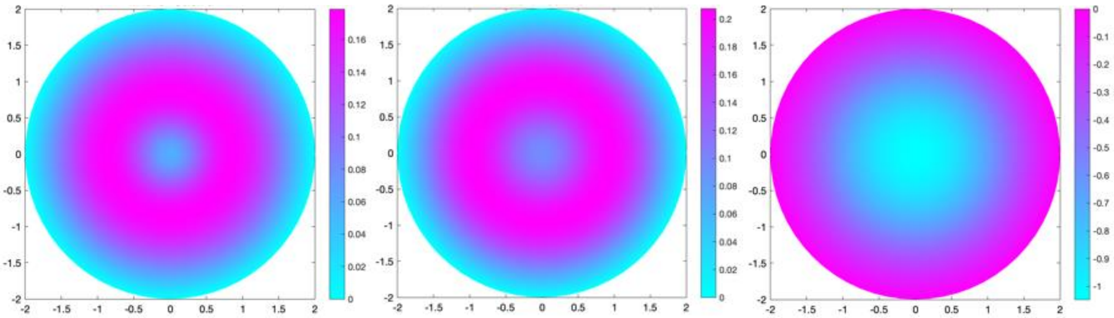

Figure 7. Poisson's equation with $f(x,y,t) = e^{-t^2}$

When $f(x,y,t)$ is dependent solely on spatial variables, for example, when $f(x,y,t) = e^{-x^2-y^2}$, the numerical solution of the wave equation is obtained with the spatially dependent forcing term:

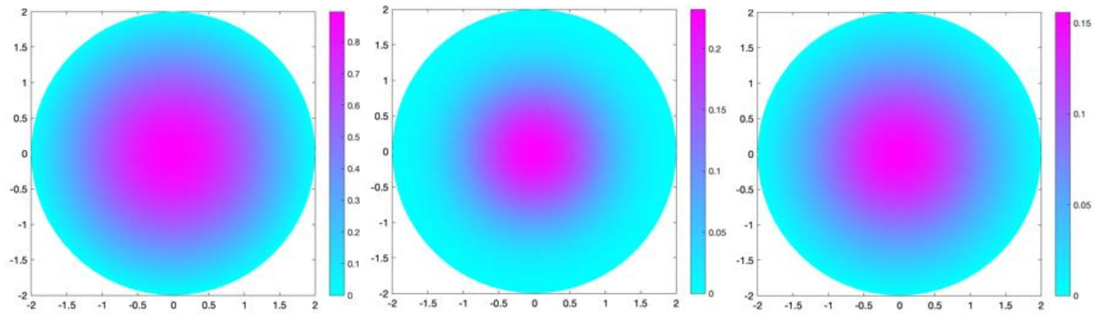

Figure 8. Poisson's equation with $f(x,y,t) = e^{-x^2-y^2}$

When $f(x,y,t)$ is a function of both space and time, for instance, when $f(x,y,t) = e^{-x^2-y^2-t^2}$, the numerical solution reflects the influence of the spatiotemporal forcing term, as computed:

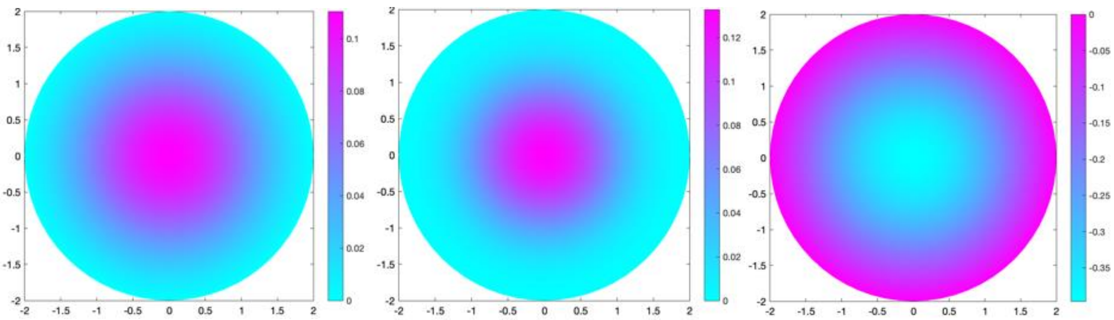

Figure 9. Poisson's equation with $f(x,y,t) = e^{-x^2-y^2-t^2}$

5. Conclusion

This paper employs the method of separation of variables to solve the analytical solutions of the irrotational Euler equations in the absence of external forces. Analytical solutions for incompressible and steady fluids have been derived, as well as numerical solutions for cases where the density is a function of time. For situations where the density is time-dependent, we assume an irrotational velocity field, leading to an equation equivalent to the Poisson equation, for which we have numerically solved a series of instances. Additionally, we have solved the numerical solutions for the two-dimensional wave equation under various external fields and have plotted the evolution of these numerical solutions over time.

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