Newton-Leibniz Formula Based on the Decomposition Method of Split Complex Zero Factors

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Abstract. Extending the classical results in real analysis to the split complex plane is currently a very active area of mathematical research, with numerous scholars conducting extensive studies in this field. This paper investigates the Newton-Leibniz formula on the split complex plane, where split complex numbers form a commutative ring with zero divisors generated by two real numbers. By overcoming the challenge posed by the presence of zero divisors in split complex numbers, this study derives the Newton-Leibniz formula on the hyperbolic plane through the factorization of zero divisors in split complex numbers. This work lays a theoretical foundation for split complex analysis and provides impetus for the development of physics grounded in the context of categorical complex analysis. This work will also establish a research foundation and inject momentum into further investigations into four-dimensional space models utilizing the split complex numbers.

Keywords: Split Complex Numbers, Partial Order, Newton-Leibniz formula.

1. Introduction

Split complex numbers are a generalization of real numbers, forming a commutative ring with zero divisors, generated by two real numbers. Lyu [1] have studied hyperbolic number sequences and proved the compression principle of hyperbolic sequences, providing a new perspective for proving the convergence of hyperbolic sequences. Wolfdieter [2] have discussed the hyperbolic complex algebraic structures in dimensions 2, 3, and 4, and provided a series of definitions for the hyperbolic vector exponential function in a standard manner. Michael [3] describe how to classify singularities in the theory of bicomplex holomorphic functions, where singularities correspond to their orders, which are hyperbolic numbers with integer components rather than real integers. Luna [4] have studied the set of bicomplex numbers from a hyperbolic modular perspective, and constructed several types of fractal sets within the four-dimensional space.

Liu [5] have studied the conditions for the validity of the Newton-Leibniz formula in infinite point sets, and proved that the Newton-Leibniz formula holds when the set of limit points is finite. Liu [6] have discussed the inherent connections among Gauss's formula, Green's formula, and Newton-Leibniz formula, pointing out that Green's formula and Newton-Leibniz formula can be viewed as Gauss's formula in one-dimensional and two-dimensional Euclidean spaces, respectively. Gao [7] have applied methods such as the Newton-Leibniz formula, starting from analyzing the inherent properties of the integrand function itself, and summarized a series of application examples with typical significance for solving generalized integrals. Tian [8] have utilized some definitions, methods for deriving composite functions, and the Newton-Leibniz formula to conduct detailed derivations of the first-order and second-order partial derivatives of the η function.

Ozturk [9] have identified a set of hyperbolic numbers for affine transformations on the Lorentz plane and provided some examples of hyperbolic fractals. Petoukhov [10] have utilized the mathematics of multidimensional hyperbolic numbers and their matrix representations to model various genetic biological systems. Alpay [11] have studied conditional hyperbolic probability, proving theorems for multiplication under hyperbolic similarity, the total probability law, and Bayes' theorem in this context. Kobayashi [12] investigated the reducibility of hyperbolic neural networks. In real-valued and complex-valued neural networks, three types of reducibility exist. Based on the decomposition of zero factors in split complex numbers, this paper simplifies the integration over the split complex plane into real integrations by decomposing the integral of split complex variables onto the zero factor axes αj_+ and βj_-. Consequently, the Newton-Leibniz formula on the split complex plane is established. Based on the above research, this paper establishes the Newton-Leibniz formula on $\mathbb{R}^{1,1}$ and derives the formula for $\mathbb{R}^{1,1}$, further laying the foundation for theoretical studies of split-complex numbers.

2. The basic fundamental of Split Complex Numbers

The algebra of split complex numbers, or double numbers

$$\mathbb{R}^{1,1} = \left\{ \xi = x + yj \mid x, y \in \mathbb{R}, j^2 = 1 \right\}. \tag{1}$$

Is the Clifford algebra $C\ell_{1,0}$ generated as an algebra over $\mathbb{R}$ by 1 and j. It is evident from the aforementioned equation that $\mathbb{R}^{1,1}$ is similar to $\mathbb{C}$, it has a different algebraic structure. Specially, $\mathbb{R}^{1,1}$ has zero divisors:

$$(1+j)(1-j) = 1 - j^2 = 0. \tag{2}$$

The definitions of two special zero factors are as follows:

$$j_+ = \frac{(1+j)}{2} \qquad j_- = \frac{(1-j)}{2}. \tag{3}$$

This paper combine j_+ and j_- to obtain a new definition for $\mathbb{R}^{1,1}$

$$\xi = uj_+ + vj_-. \tag{4}$$

By calculating the above equation, we find that squaring j_+ and j_- results in them remaining the same

$$j_+{}^2 = j_+ \qquad j_-{}^2 = j_-. \tag{5}$$

Hence, based on the above formula, multiplication is simplified under this new definition, yielding the following equations

$$(u_1 j_+ + v_2 j_-)(u_2 j_+ + v_2 j_-) = u_1 u_2 j_+ + v_1 v_2 j_-. \tag{6}$$

Through deduction, we can also derive very useful operational properties that general operations on $\mathbb{R}$ do not possess

$$\frac{aj_+ + bj_-}{cj_+ + dj_-} = \frac{a}{c} j_+ + \frac{b}{d} j_- \left(c, d \neq 0\right). \tag{7}$$

From the above discussion, it is evident that the zero divisors can be succinctly expressed in the following form

$$\xi = \alpha j_+ \qquad \text{or} \qquad \psi = \beta j_- \qquad (\alpha, \beta \in \mathbb{R}). \tag{8}$$

These elements are referred to as the light cone, denoted by L, which is the union of two one-dimensional subspaces

$$L_+ = \{\alpha j_+ \mid \alpha \in \mathbb{R}\} \qquad \text{and} \qquad L_- = \{\beta j_- \mid \beta \in \mathbb{R}\}. \tag{9}$$

Except for the zero divisors on the l and o axes, y and r exhibit excellent properties, including invertibility

$$\xi^{-1} = u^{-1} j_+ + v^{-1} j_-. \tag{10}$$

This paper establish a two-dimensional Cartesian coordinate system with x as the horizontal axis and yj as the vertical axis. Below is the legend for this coordinate diagram.

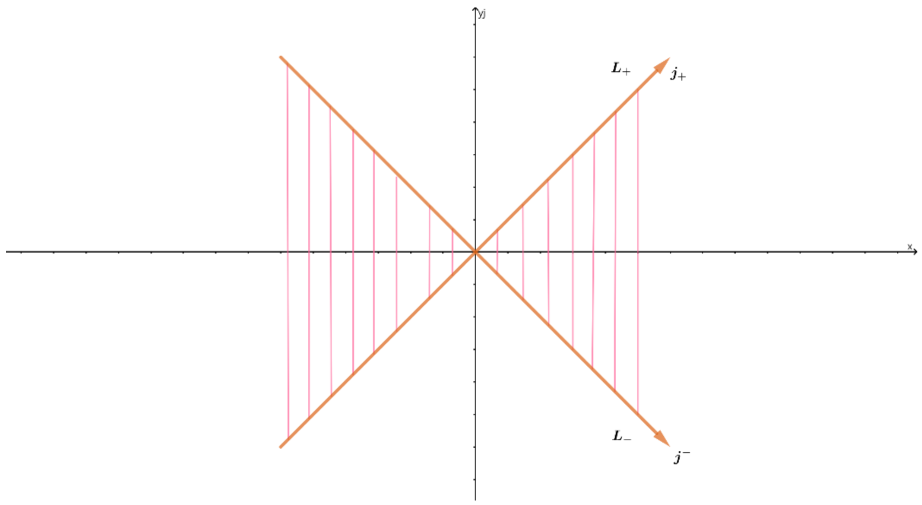

Figure 1. The positive and negative Split Complex Numbers

Based on the properties of j_+ and j_-, we add two new axes, corresponding to L_+ and L_-, respectively.

In the above diagram, this paper define the shaded region on the left as $\mathbb{R}_-^{1.1}$ and the shaded region on the right as $\mathbb{R}_+^{1.1}$. The geometric representations of $\mathbb{R}_+^{1.1}$ and $\mathbb{R}_-^{1.1}$ are shown in Figure 1.

$$\mathbb{R}_+^{1.1} = \left\{ \xi = a_1 j_+ + b_1 j_- \mid a_1, b_1 \geq 0 \right\} \qquad \mathbb{R}_-^{1.1} = \left\{ \psi = a_2 j_+ + b_2 j_- \mid a_2, b_2 \leq 0 \right\}. \tag{11}$$

This paper cannot define the concept of order on $\xi = x + yj$, but after introducing j_+ and j_- and defining $\delta = u_1 j_+ + v_1 j_-$ and $\omega = u_2 j_+ + v_2 j_- \left(u_1, u_2, v_1, v_2 \in \mathbb{R} \right)$, we can define the concept of order

$$\delta \preceq \omega \text{ iff } \omega - \xi \in \mathbb{R}_+^{1.1} \qquad \omega \preceq \xi \text{ iff } \omega - \xi \in \mathbb{R}_-^{1.1}. \tag{12}$$

This paper represent the concept of order in the diagram. The partial order relation is illustrated in Figure 2.

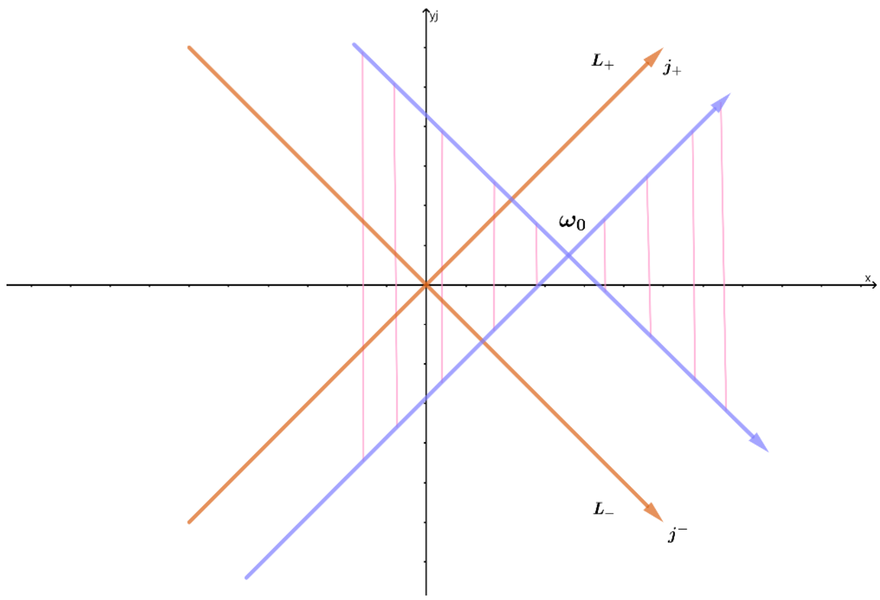

Figure 2. The concept of partial order on $\mathbb{R}^{1.1}$

It can be seen that the entire plane is divided into four quadrants, and a binary relation is introduced. This relation is reflexive, transitive, and antisymmetric, defining a partial order relation on $\mathbb{R}^{1.1}$.

The concept of an interval in $\mathbb{R}$ can similarly be introduced into $\mathbb{R}^{1.1}$. The geometric interpretation of an interval on the $\mathbb{R}^{1.1}$ is shown in Figure 3

$$\left[\delta,\omega\right]_{\mathbb{R}^{1.1}} = \left\{u \in \mathbb{R}^{1.1} \mid \delta \preceq u \preceq \omega\right\}. \tag{13}$$

Equivalently, $u = a\ j_+ + b\ j_- \in \left[\delta,\omega\right]_{\mathbb{R}^{1.1}}$ iff

$$u_1 \le a \le u_2 \ \text{ and } \ v_1 \le b \le v_2. \tag{14}$$

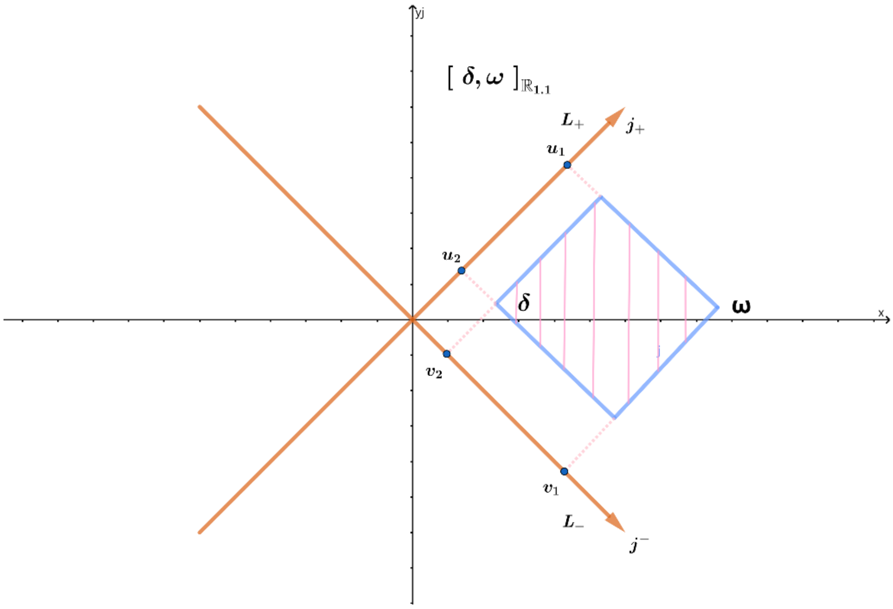

Figure 3. The hyperbolic interval $\left[\delta,\omega\right]_{\mathbb{R}^{1.1}}$

This paper introduce the concept of modulus and define the k-modulus of the element ξ on $\mathbb{R}^{1.1}$

$$\xi = uj_+ + vj_- \quad |\xi|_k = |u|\, j_+ + |v|\, j_-. \tag{15}$$

Now, let's define functions of a split-complex variable

$$f : U \subseteq \mathbb{R}^{1.1} \to \mathbb{R}^{1.1}. \tag{16}$$

And those functions may be written:

$$f(\xi) = f_1(x,y) + f_2(x,y). \tag{17}$$

Below, this paper provide the definition of the derivative on $\mathbb{R}^{1.1}$. Clearly, in $\mathbb{R}^{1.1}$, zero divisors on the two axes do not have derivatives

$$\lim_{\substack{\xi\to\xi_0 \\ \xi\notin L}} \frac{f(\xi) - f(\xi_0)}{\xi - \xi_0} = f'(\xi_0). \tag{18}$$

f can be expressed as a split-complex valued function of a split-complex variable. This paper say f is $C\ell_{1.0}$ meromorphic if

$$f(\xi) = f_1(u)\, j_+ + f_2(v)\, j_-. \tag{19}$$

And f_1 and f_2 are real differentiable functions that satisfy

$$f'(\xi) = f_1'(u)\, j_+ + f_2'(v)\, j_-. \tag{20}$$

Finally, this paper define the concept of integration. Integration in $\mathbb{R}^{1.1}$ involves transforming a surface integral into a line integral along L_+ and L_-. We define it as follows

$$I=\int_H f(\xi)\otimes d\xi \qquad d\xi=(dj_+,dj_-). \tag{21}$$

This paper define the mapping $f:[\delta,\omega]_{\mathbb{R}^{1.1}}\to\mathbb{R}^{1.1}$

$$\int_{[\delta,\omega]_{\mathbb{R}^{1.1}}} f(\xi)\otimes d\xi=\left\{\int_{u_1}^{u_2} f_1(u)du\right\}j_+ +\left\{\int_{v_1}^{v_2} f_2(v)dv\right\}j_-. \tag{22}$$

3. Results

Theorem 3.1 [Newton-Leibniz Formula on $\mathbb{R}^{1.1}$]: The geometric meaning of Newton-Leibniz formula is shown in Figure 4. If f is $C\ell_{1,0}$ differentiable on the hyperbolic interval $[\delta,\omega]_{\mathbb{R}^{1.1}}$, then there exists a split complex variable function F, for which

$$F'(\xi)=F_1'(u)j_+ +F_2'(v)j_-=f(\xi) \qquad \xi\in[\delta,\omega]_{\mathbb{R}^{1.1}}. \tag{23}$$

Then f is integrable on the interval $[\delta,\omega]_{\mathbb{R}^{1.1}}$, and satisfy the following formula:

$$\int_{[\delta,\omega]_{\mathbb{R}^{1.1}}} f(\xi)\otimes d\xi=F(\omega)-F(\delta) \tag{24}$$

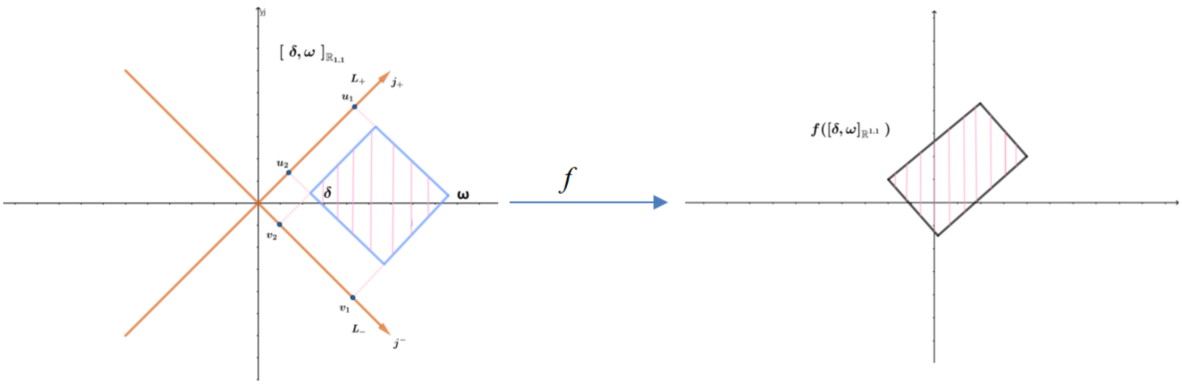

Figure 4. The geometric meaning of Newton-Leibniz formula

Proof: First, let

$$\delta=u_1j_+ +v_1j_- \qquad \omega=u_2j_+ +v_2j_- \tag{25}$$

Then

$$F(\delta)=F_1(u_1)j_+ +F_2(v_1)j_- \qquad F(\omega)=F_1(u_2)j_+ +F_2(v_2)j_-. \tag{26}$$

It follows from the given conditions and (23) that

$$\begin{aligned}&\int_{[\delta,\omega]_{\mathbb{R}^{1.1}}} f(\xi)\otimes d\xi\\ &=\int_{[\delta,\omega]_{\mathbb{R}^{1.1}}} F'(\xi)\otimes d\xi\\ &=\int_{[\delta,\omega]_{\mathbb{R}^{1.1}}}\left(F_1'(u)j_+ +F_2'(v)j_-\right)\otimes(j_+du+j_-dv)\\ &=j_+\int_{u_1}^{u_2}F_1'(u)du+j_-\int_{v_1}^{v_2}F_2'(v)dv\end{aligned} \tag{27}$$

According to the Newton-Leibniz on the real numbers, it follows that

$$\int_{u_1}^{u_2} F_1'(u)du = F_1(u_2) - F_1(u_1) \quad \int_{v_1}^{v_2} F_2'(v)dv = F_2(v_2) - F_2(v_1) \tag{28}$$

Combine with examples (26) and (27), it follows that

$$\int_{[\delta,\omega]_{\mathbb{R}^{1,1}}} f(\xi) \otimes d\xi = j_+ \left(F(u_1) - F(u_2)\right) + j_- \left(F(v_1) - F(v_2)\right) = F(\omega) - F(\delta). \tag{29}$$

The proof is completed. The proof here is based on the decomposition of split complex variable functions with respect to their zero factors. Through this decomposition, the integral over the split complex plane is decomposed onto the two zero factor axes, thereby simplifying the integral operation.

4. Conclusion

This paper extends the Newton-Leibniz formula in real analysis to obtain the Newton-Leibniz formula on the split complex plane. This research methodology is currently very active, with numerous scholars having achieved new research findings in this area. Since split complex numbers are commutative rings with zero divisors generated by two real numbers, this paper overcomes the difficulties caused by zero divisors based on the decomposition properties of zero divisors of split complex numbers and simplifies the integral over the split complex plane by decomposing the integral through the decomposition of zero factors, thereby deriving the Newton-Leibniz formula on the split complex plane. This will further provide a theoretical basis for the development of split complex analysis and inject vitality into the development of split complex numbers in physics. Moreover, it will enrich the field of split complex analysis.

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