The Properties of Green's Function and Variation of Constants in Damped Oscillations Using Numerical Approach

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Abstract. Contemporarily, Green function is considered as a basic mathematical formular in various applications in advanced physics. On this basis, this study analyzes the efficacy and precision of Green's Function as well as the Variation of Constants (VoC) techniques in addressing damped oscillation issues, particularly in systems affected by an impulsive external force, exemplified by a Dirac delta function. To be specific, the analysis examines the duration necessary for a damped oscillator to be deemed at rest, contrasting the efficacy of Green's Function and VoC in forecasting critical moments and addressing numerical difficulties spanning underdamped, overdamped, and severely damped conditions. According to the analysis, numerical simulations were performed based on Python to evaluate the convergence, stability, and computational efficiency of each method. The results highlight the comparative advantages as well as disadvantages of each method, providing important insights for applications in oscillatory systems where accurate and prompt response predictions are critical.

Keywords: Damped oscillation; Green's Function; variation of constants; numerical analysis.

1. Introduction

As a matter of fact, damped oscillations exhibit a decline in oscillatory motion that follows an exponential trajectory, potentially converging towards zero without ever attaining it within a finite duration. This notion corresponds with the principles of Green's Functions, which are extensively utilized in oscillatory and damping analysis [1]. In academic settings, particularly in physics courses, these ideas are essential for comprehending the mathematical foundations of oscillatory motion and damping effects [2, 3]. This study examines Green's Functions in relation to a damped harmonic oscillator influenced by an impulsive external force, akin to the situations addressed in the Physics 228 course [2].

2. Methodology

2.1. Deriving Green's Function

Green's Function proficiently addresses second-order differential equations in oscillatory systems with diverse damping, encapsulating forced oscillation dynamics under varying parameters [4, 5]. A sketch is shown in Fig. 1. Initially, consider an oscillator with a mass that adheres to the equation of motion:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F(t)}{m} \tag{1}$$

where $2\gamma\dot{x}$ is a damping term, γ is the damping coefficient, ω_0 is the natural frequency of this oscillator and $F(t)$ is an external driving force. To make this equation physical, $\gamma > 0$ and $\omega_0 > 0$ are required. Applying Fourier transforms and inverse transformations to solve for $G(t)$, Green's Function is derived as

$$G(t,t') = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{e^{i\omega(t-t')}}{(\omega-\omega_+)(\omega-\omega_-)} \tag{2}$$

where $\omega_+ = i\gamma + \sqrt{\omega_0^2 - \gamma^2}$ and $\omega_- = i\gamma - \sqrt{\omega_0^2 - \gamma^2}$, assuming $\gamma \neq \omega_0$. Considering the residue theorem that:

$$\oint_\gamma f(z)dz = 2\pi i \sum \text{Res}(f, a_k) \tag{3}$$

where γ is a positively oriented simple closed curve, a_k is 1 if it is in the interior of γ and 0 if not.

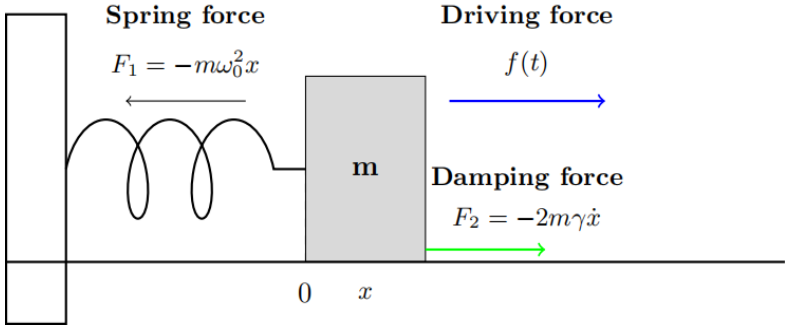

Fig. 1 Description of the Damped Oscillator.

Before solving, consider the physical conditions of the oscillator's Green's function. Green's function solves the equation of motion with an external driving force as a δ function, which is zero except at $t = 0$. Physically, the oscillator remains static before the pulse at $t = 0$, after which it starts moving. Thus, a step function Θ is introduced:

$$\Theta(\tau) = \begin{cases} 1, & \text{for } \tau \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Finally, combing all possible values of ω, the solution of Green's function will be

$$G(t,t') = \Theta(t-t')e^{-\gamma(t-t')} \times \begin{cases} \dfrac{1}{\sqrt{\omega_0^2 - \gamma^2}} \sin\left(\sqrt{\omega_0^2 - \gamma^2}(t-t')\right), & \gamma < \omega_0 \\ (t-t'), & \gamma = \omega_0 \\ \dfrac{1}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\left(\sqrt{\gamma^2 - \omega_0^2}(t-t')\right), & \gamma > \omega_0 \end{cases} \tag{5}$$

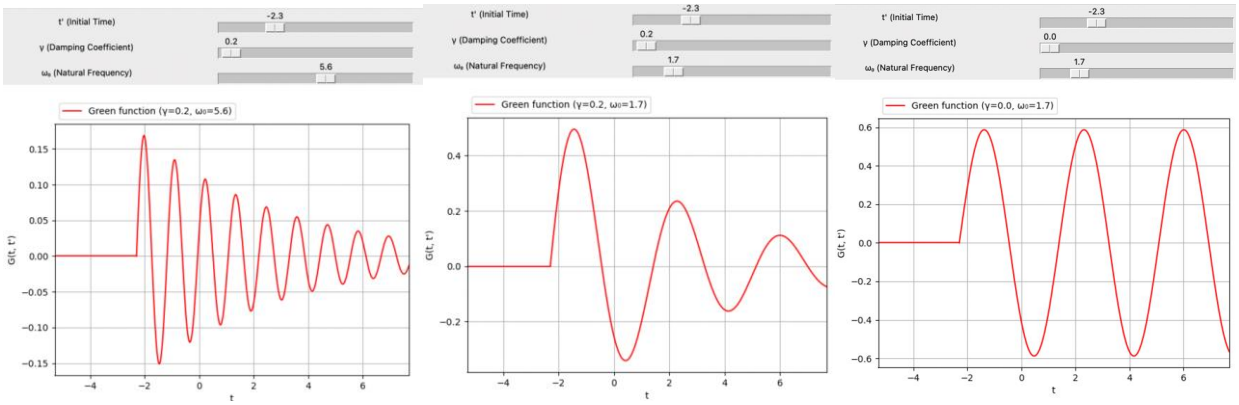

Fig. 2 Under-damped conditions: damping with higher frequency, lower frequency and without damping (from left to the right).

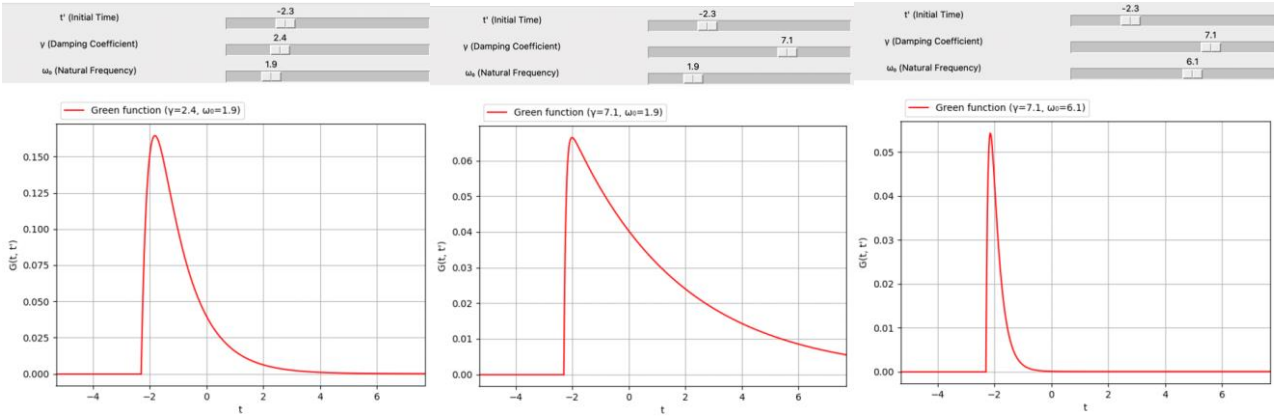

Fig. 3 Over-damped conditions: lower γ & lower ω_0; higher γ & lower ω_0 and (from left to the right).

Straightforwardly, t' defines the initial time of the motion of this oscillator. Before t', it should be static. Depending on the relative values of γ and ω_0, Green's function will have a linear or a sine or a hyperbolic sine function of t. To study the behaviour of Green's function, use Python to plot figures showing the motion of this oscillator in different conditions. First, when $\gamma < \omega_0$, Fig. 2 shows cases for under-damped conditions Then, when $\gamma > \omega_0$, one plots three case as shown in Fig. 3. Finally, one plots the case for $\gamma = \omega_0$ as shown in Fig. 4. Before the initial time, the oscillator is static. After applying a pulse, the motion depends on the values of damping coefficient γ and natural frequency ω_0. In the case of under damping, the oscillator will oscillate and passe through zero multiple times with an exponentially decreasing amplitude. However, for critical damping and over damping, the oscillator will not reach but keep approaching zero and the decreasing velocity depends on the combination of γ and ω_0.

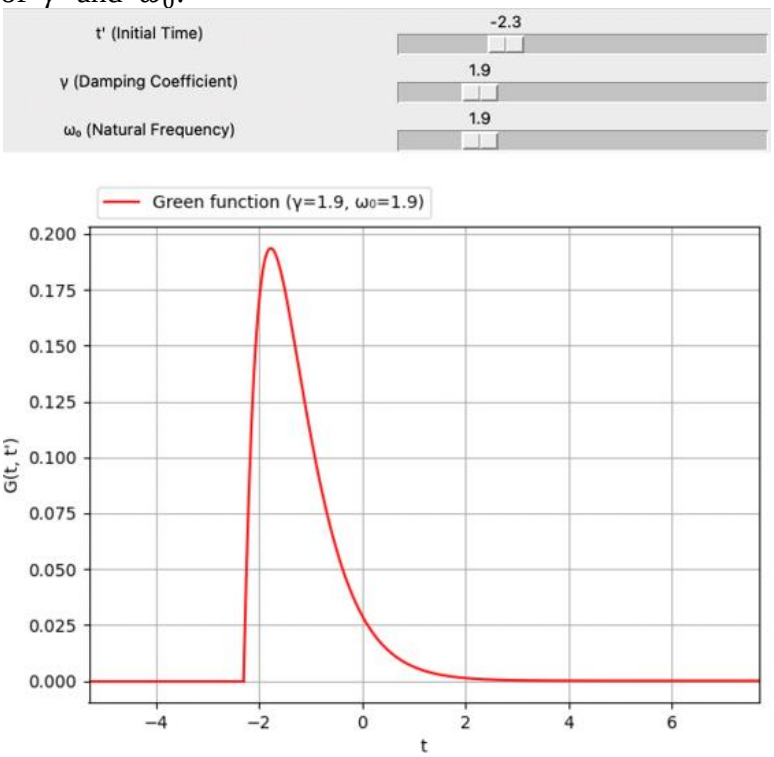

Fig. 4 Damping coefficient equals frequency.

2.2. The Method of Variation of Constants

VoC solves non-homogeneous differential equations in systems with time-dependent forces, complementing Green's Function in forced oscillations [6-8]. The complete solution integrates the homogeneous and particular components of the differential equation is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + u_1(x) y_1(x) + u_2(x) y_2(x) \tag{6}$$

For the second-order non-homogeneous differential equation given above, first need to find the general solution of the corresponding homogeneous equation:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{7}$$

The two solutions of the characteristic equation for this expression are:

$$r_{\pm} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{8}$$

Depending on the value of γ relative to ω_0, there are three different cases, which are described as follows:

$$x(t) = \begin{cases} \text{Underdamped:} & e^{-\gamma t}\left(C_1 \cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right) + C_2 \sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right)\right) \\ & - \int \frac{\sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right)}{\sqrt{\omega_0^2 - \gamma^2}} \frac{F(t)}{m} e^{\gamma t}\, dt \cdot e^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right) \\ & + \int \frac{\cos\left(\sqrt{\omega_0^2 - \gamma^2}t\right)}{\sqrt{\omega_0^2 - \gamma^2}} \frac{F(t)}{m} e^{\gamma t}\, dt \cdot e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}t\right) \\ \text{Overdamped:} & C_1 e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t} + C_2 e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t} \\ & + \int \frac{e^{\left(\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t}}{2\sqrt{\gamma^2 - \omega_0^2}} \frac{F(t)}{m}\, dt \cdot e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t} \\ & - \int \frac{e^{\left(\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)t}}{2\sqrt{\gamma^2 - \omega_0^2}} \frac{F(t)}{m}\, dt \cdot e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)t} \\ \text{Criticallydamped:} & e^{-\gamma t}(C_1 + C_2 t) - \int t e^{\gamma t} \frac{F(t)}{m}\, dt \cdot e^{-\gamma t} \\ & + \int e^{\gamma t} \frac{F(t)}{m}\, dt \cdot t e^{-\gamma t}. \end{cases} \tag{9}$$

Then, one fixes the external force $F(t)$ as a transient pulse (δ-function), just as the way when calculating Green's function. The solution of the constant variation method can well represent the behavior of motion at time t, so it is not necessary to add the Heaviside function. Since the focus is solely on the behavior of the system after external forces are applied (i.e., $t \geq t'$), the solution of the variation of constants method for $t < t'$, is disregarded, yielding the final results.

$$x(t) = \begin{cases} e^{-\gamma(t-t')}\left(C_1 \cos\left(\sqrt{\omega_0^2 - \gamma^2}(t-t')\right) + C_2 \sin\left(\sqrt{\omega_0^2 - \gamma^2}(t-t')\right)\right), & \gamma < \omega_0 \\ C_1 e^{\left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)(t-t')} + C_2 e^{\left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)(t-t')}, & \gamma > \omega_0 \\ e^{-\gamma(t-t')}\left(C_1 + C_2(t-t')\right), & \gamma = \omega_0 \end{cases} \tag{10}$$

2.3. Quantifying System Quiescence through Deviation Ratios

Green's Function shows oscillations approaching zero over time, with deviation ratios α representing effective system rest times by comparing the maximum deviation to the value at time t.

For example, the time required for the oscillator to reach 0.001 of its maximum deviation may be estimated. First, the maximum deviation is calculated. From the figures above, it is evident that Green's Function quickly reaches its maximum and then declines over time. For all three cases, values are calculated between t' and $t' + 1/\gamma$ from which the maximum value G_m is derived. Newton's Method iteratively refines approximations to identify root values where deviation ratios α indicate effective rest. The iterative formula for Newton's method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{11}$$

where x_n is the current approximation; $f(x_n)$ is the value of the function at x_n; $f'(x_n)$ is the derivative of the function at x_n; x_{n+1} is the next, improved approximation. Using Newton's Method, one iteratively refines the initial guess x_0 to find the root x_n where α smaller enough (e.g., $\alpha \approx 0.2$). Convergence is achieved when $|x_{n+1} - x_n|$ falls within a defined tolerance. For the overdamped and critically damped cases, Green's Function exhibits a monotonically decreasing behavior. By setting $G(t, t') = \alpha G_m$, Newton's method can be directly applied to determine the time at which Green's Function reaches the target value. However, in the underdamped case, where $G(t, t')$ oscillates periodically, $G(t, t') = \alpha G_m$ may have multiple roots, making Newton's method less effective. To address this, $G(t, t')$ must first be constrained to a narrower range, and an appropriate starting point should be chosen. The exponential part of Green's Function can be defined as a reference function:

$$R(t, t') = \frac{e^{-\gamma(t-t')}}{\sqrt{\omega_0^2 - \gamma^2}} \tag{12}$$

The range of the true root t_α is estimated by calculating $R(t_0, t') = \alpha G_m$. After t_0, $G(t, t')$ remains definitively smaller than αG_m. The closest local minimum t_1 and local maximum t_2 are then identified, with the cases depicted in Fig. 5. When $t_0 > t_1$, the true root is located between t_2 and t_1. Conversely, when $t_0 < t_1$, the true root is located between t_2 and t_0. The starting point is then set as the midpoint of this range, ensuring that Newton's method operates effectively under these conditions.

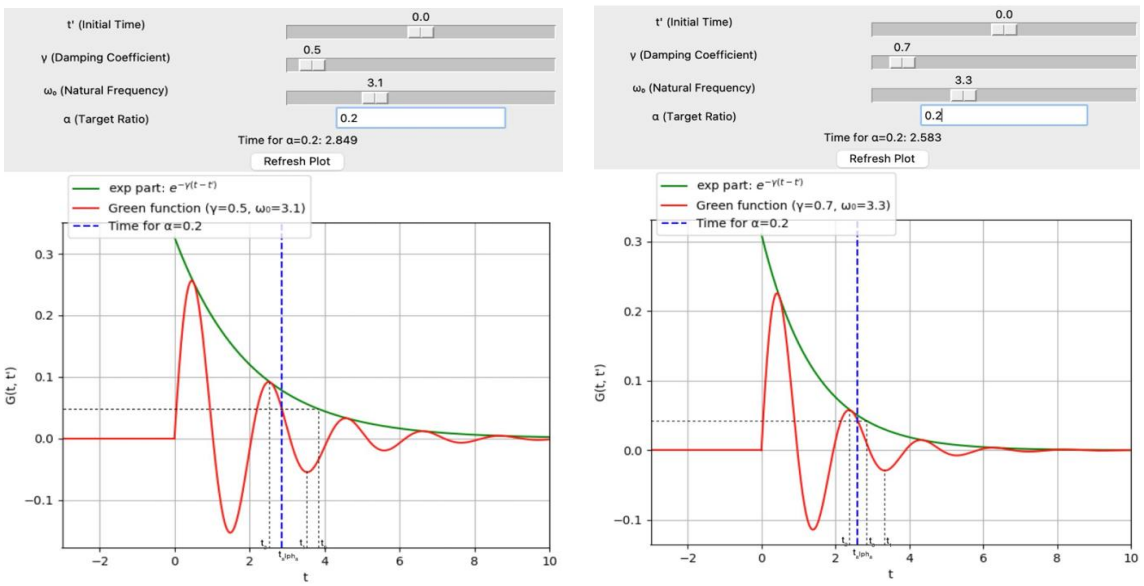

Fig. 5 Root range of the under-damped case: from case of $t_0 > t_1$ and (from the left to the right).

After establishing Newton's method for the Green's function, attention shifts to applying this numerical technique to the solution obtained via the method of Variation of Constants. One defines the function $f(t)$ for which the roots are sought, satisfying $x_p(t) = \alpha \max(x_p)$, where α is the small ratio previously defined (e.g., 0.2). This involves:

Calculating $\max(x_p)$: Find the maximum response of the particular solution over an appropriate time interval following the impulse.

Defining $f(t)$: Set up the function $f(t) = x_p(t) - \alpha\max(x_p)$.

Applying Newton's method: Iterate using the Newton-Raphson formula to find the time t at which $f(t) = 0$, indicating that the response has decayed to α times its maximum value.

The variables are controlled, applying the same operations as those used for the Green's function above, though the analytic solution takes a different form. This approach allows progression toward the final comparative analysis.

3. Quantitative Comparison and Precision Analysis

Performance differences reflect the complexity of the numerical methods. With varied solver tolerance and step size, VoC consistently showed lower mean errors than Green's Function, particularly in highly damped systems. Thus, method selection should consider damping characteristics and computational constraints. While Green's Function captures impulse responses well in oscillatory systems, it incurs larger errors in heavily damped regimes.

3.1. Numerical Convergence and Stability

Numerical methods verify stability and convergence for Green's Function and VoC under different damping regimes [5, 10]. To evaluate the numerical convergence and stability of the Green's Function and Variation of Constants (VoC) methods, tests were conducted with varying step sizes and solver tolerances. The objective was to observe each method's behavior under increasingly precise constraints and to track the evolution of errors with finer time-step resolutions. For each damping regime—underdamped, overdamped, and critically damped—Python's built-in solver, solve_ivp (based on the Runge-Kutta method), was used as a reference. The results, including execution times, mean absolute errors (MAE), and maximum errors, were computed as the number of time steps increased. Following error metrics are introduced. MAE quantifies the average deviation of the numerical solution from the exact or reference solution:

$$\mathrm{MAE} = \frac{1}{N}\sum_{i=1}^{N} |x_{\mathrm{num}}(t_i) - x_{\mathrm{exact}}(t_i)| \tag{13}$$

Here, $x_{\mathrm{num}}(t_i)$ is the numerical solution at time t_i, and $x_{\mathrm{exact}}(t_i)$ is the exact solution. N is the number of sampled points. Maximum error quantifies the worst deviation between numerical and exact solutions.

$$\mathrm{MaxError} = \max_i |x_{\mathrm{num}}(t_i) - x_{\mathrm{exact}}(t_i)| \tag{14}$$

This metric helps in understanding the largest discrepancy between the two solutions over the time interval. The results are listed in Table 1.

Table 1. Convergence Results for Different Step Sizes and Solver Tolerances

Damping Regime	Solver Tolerance	Green's Time (s)	VoC Time (s)
Underdamped	10^{-3}	0.0007	0.0005
	10^{-6}	0.0006	0.0005
	10^{-9}	0.0006	0.0005
Overdamped	10^{-3}	0.0006	0.0005
	10^{-6}	0.0006	0.0005
	10^{-9}	0.0006	0.0005
Critically Damped	10^{-3}	0.0002	0.0002
	10^{-6}	0.0002	0.0002
	10^{-9}	0.0003	0.0002

3.2. Computational Efficiency

Green's Function generally outperforms VoC in detecting transitions in critically and overdamped systems (Baghai-Wadji, 2000; Mohammed & Al-Nuaimi, 2023) [9-11]. Execution times for both methods were assessed across damping regimes, solver tolerances, and step sizes to evaluate computational performance (Table 1). Green's Function typically performs faster in critically and overdamped cases, but at higher solver tolerances, execution times converge, indicating both methods are effective. In underdamped regimes, Green's Function requires slightly more time for larger steps and smaller tolerances due to the complexity of oscillatory behavior. Newton's method was used to determine the critical time t_α, when the response reaches a fraction α of the maximum deviation, for both solutions.

3.3. Accuracy in Finding Critical Times (like t_α)

Newton's method effectively estimates critical times t_α, across both Green's Function and VoC, yielding consistent results. Results for $\alpha = 0.2$ are as follows:

Underdamped case: t_α for Green's Function is 9.7546×10^{-1}, and t_α for VoC is 9.7546×10^{-1}.

Overdamped case: t_α for Green's Function is 7.0450×10^{0}, and for VoC, it is 7.0450×10^{0}.

Critically Damped case: t_α for Green's Function is 2.6629×10^{0}, and for VoC, it is 2.6629×10^{0}.

In all three cases, Green's Function consistently yields t_α avalues similar to VoC, showing both methods effectively detect the time when the response reaches a specific fraction of the maximum. This similarity indicates that, with high precision, VoC can be as reliable as Green's Function for determining critical times across damping regimes. While both methods accurately capture critical response times, differences in computational efficiency may make one preferable depending on specific requirements.

3.4. Error Analysis

MAE and maximum error analysis show lower errors for VoC, especially in high-damping conditions. The results are listed in Table 2. Performance differences reflect the complexity of the numerical methods. As solver tolerance and step size fluctuated, VoC consistently exhibited fewer mean errors compared to Green's Function, particularly in systems with strong damping. These findings indicate that the choice of approach should take into account damping characteristics and computing limitations. Although Green's Function is proficient in oscillatory systems for obtaining impulse responses, it results in greater inaccuracies in substantially damped conditions.

Table 2. Error Analysis for Different Damping Regimes

Damping Regime	Mean Absolute Error (Green's)	Mean Absolute Error (VoC)	Maximum Error (Green's)	Maximum Error (VoC)
Underdamped	2.5559×10^{-1}	1.6942×10^{-1}	1.2356×10^{-1}	1.0517×10^{0}
Overdamped	4.4791×10^{-1}	1.7730×10^{-1}	1.1188×10^{-1}	9.8101×10^{-1}
Critically Damped	5.2772×10^{-1}	3.6959×10^{-1}	9.8439×10^{-1}	8.7414×10^{-1}

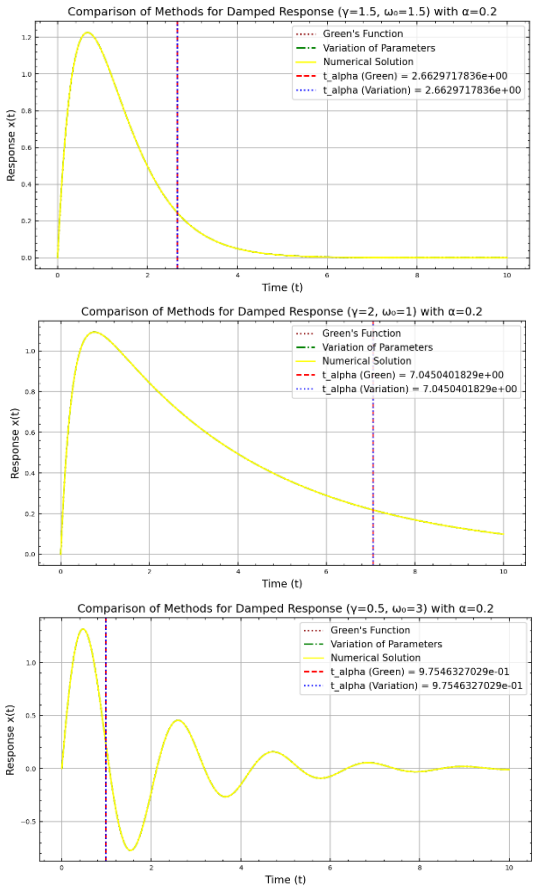

Fig. 6 Comparative Plots for Green's Function and VoC Across Damping Regimes: Critically Damped Case; Underdamped Case and Overdamped Case (from left to the right).

3.5. Behavior Across Three Damped Cases

Green's Function and VoC demonstrate advantages in various damping circumstances, with VoC offering improved stability in high-damping systems [7]. The behavior of the Green's Function and VoC solutions was examined using graphical comparisons spanning underdamped, overdamped, and severely damped regimes. Fig. 6 illustrates the response curves for each approach over various damping settings, with critical times denoted by vertical lines.

In the underdamped scenario, both Green's Function and VoC ascertain the critical time t_α with comparable values; however, Green's Function exhibits reduced numerical error, more effectively representing oscillatory activity. In the overdamped scenario, Green's Function exhibits somewhat greater deviations than VoC, which offers a more precise response. In the critically damped scenario, both methodologies exhibit analogous tendencies; however, Green's Function presents marginally greater mistakes, whereas VoC displays superior stability, rendering it a more dependable option.

4. Conclusion

To sum up, Green's Function is an effective instrument for transient analysis in damped systems, particularly advantageous for early response identification when computer resources permit its elevated costs. Although optimal for monitoring oscillatory behavior, its long-term accuracy may be inferior in significantly damped situations compared to VoC. For the advantages, Green's Function finds critical moments t_α sooner in underdamped regimes, suitable for applications requiring swift response. Beisdes, Green's Function precisely tracks progressive decays, effective in systems exhibiting substantial damping. Although there is increased numerical error in underdamped scenarios, Green's Function proficiently delineates oscillatory patterns. Nevertheless, there are some limitations. For error accumulation, Green's Function may accrue inaccuracies in slow-decay systems,

diminishing precision over prolonged durations. Besides, performance fluctuates depending on numerical techniques, such as Newton's method in oscillatory contexts. Moreover, Green's Function necessitates greater resources in underdamped conditions, affecting real-time application. Outcomes fluctuate according to initial conditions, particularly in underdamped systems.

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