Vanishing and Residue of Pontryagin Classes on Singular Foliations

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Abstract. In this paper, we study the vanishing and residue of the Pontryagin classes on singular foliations on smooth manifolds. Specifically, we extend the Bott Vanishing Theorem to singular foliations that admit resolutions by vector bundles, which can be represented by L_∞-algebroids, and subsequently prove the Residue Existence Theorem for this type of singular foliations. Our results provide a way of computing the characteristic classes, particularly the Pontryagin classes on smooth manifolds using cohesive modules developed by J. Block. This approach potentially offers a new path in studying the differential geometry and topology of singular foliations beyond the traditional operator algebraic approach.

Keywords: Characteristic classes, singular foliations, manifold topology, higher structures.

1. Introduction

Connected with topology, differential geometry, and algebraic geometry, the residues associated with the singularities of foliations have been a rich field of research as it encodes the intrinsic topology of the manifolds. In Baum and Botts work [9], the residues of holomorphic foliations arelinked to polynomials in the Chern classes of the normal bundles and the tools to study the structures of singularities in complex manifolds are developed. Further, this theory has been extended to Riemannian foliations in Lazarov and Pasternak's work [21] and generalized to projective foliations by S. Nishikawa [24]. Meanwhile, the concept of the Pontryagin polynomial residues off isolated singulars in real foliations has been introduced in Schweitzer and Whitman's work [28], applying residue theory beyond holomorphic foliations. However, their research was restricted to isolated singularities, which is a rather strong restriction. This instinctively leads to the question whether this restriction can be removed to achieve a more general result. This research aims to remove this restriction to provide a more comprehensive residue theory for real smooth foliations with both isolated and nonisolated singularities. This study offers new insights into the topology of foliations with more complex singular structures and provides a more flexible toolkit for examining the topological and geometric characteristics of real foliations.

There are two key objects we are studying in this research: the Pontryagin classes andsingular foliations. First, the Pontryagin classes are a type of characteristic classes, serving as a crucial topological invariant for smooth manifolds. They are useful in providing insights into the topology and geometry of vector bundles. Moreover, singular foliations are a generalization of regular foliations which allow singularities and thus their complex ities result in challenges in the interpretation of construction of differential geometry and topology such as the Pontryagin classes. In this paper, we focus on studying the conditions under which thePontryagin classes vanish,yielding a residue theorem on singular foliations with certain types of singularities and homological properties.

Since traditional tools in differential geometry often avoid singularities, this study aimsto approach the goal by employing higher categorical and homotopical methods, i.e. Higher *differential geometry*, including $\dot{L}_\infty$ -algebroids [25] [26] [16] [20] and cohesive modules [12] [3]. By generalizing theBott Vanishing Theorem, we show that the Pontryagin classes vanish outside the singular set of the singular foliations which can be resolved by vector bundles, thereby establishing a new residue existence theorem. Note that although more of the tools we used could be formulated in terms of ∞ -categories, we would not develop a theory using ∞ -categorical language in this paper to keep the content more concrete and computable.

This paper is structured as follows:

(1) Introduce the key tools and objects L_∞-algebroids, cohesive modules, singular foliations, and Ponrtyagin classes.

(2) Extend the definition of Pontryagin classes to cohesive modules and foliations represented by L_∞ -algebroids.

(3) Generalize theBott Vanishing theorem and residue existence theorem to singular foliations with resolutions.

(4) Discuss the implications of our results and outline possible future works.

2. Pontryagin Classes on Foliations

Differential geometry provides the necessary tools and language to precisely describe and analyze the geometric properties of spaces, particularly those that are smoothly curved like manifolds. Understanding concepts such as differentiable manifolds, vector bundles, connections, and curvature is crucial for grasping the behavior of singular foliations within these spaces. After equipping ourselves with the analytical prowess required to tackle the complexities of characteristic classes, we can form a deeper comprehension of their significance and applications in the study of singular foliations.

2.1. Foliations.

A foliation provides a way to decompose a manifold into leaves, which are disjoint immersed sub manifolds of the same dimension. These leaves fit together smoothly, resembling a stack of pages in a book or the layers of an onion. More formally, we have the following definition:

Definition 2.1 ([22). Let M be a smooth manifold of dimension n.A (regular) foliation $\mathcal{F}$ of codimension q on M is described in the following equivalent ways:

(1) *Foliation Atlas:* A foliation atlas $\{\phi_i: U_i \to \mathbb{R}^{n-q} \times \mathbb{R}^q\}$ of M where the transition functions $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are of the form

$$\phi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)) \tag{1}$$

With respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$ .Here, $x \in \mathbb{R}^{n-q}$ represents coordinates along the leaves (leaf directions) and $y \in \mathbb{R}^q$ represents coordinates transverse to the leaves (transverse directions). Each connected component of. $\phi_i^{-1}(\mathbf{R}^{n-q} \times \{y\})$, for a fixed $y \in \mathbb{R}^q$, is called a plaque. Plaques glue together smoothly to form the leaves of the foliation, which are immersed sub manifolds of dimension $n - q$

(2) *Submersions and Haefliger Cocycle:* An open cover $\{\mathrm{U_i}\}$ of M with submersions $\mathrm{S_i}$ $\mathrm{U_i} \to \mathbb{R}^\mathrm{q}$ such that on overlaps $\mathrm{U_i} \cap \mathrm{U_j}$ there exist diffeomorphisms

$$\gamma_{ij}: s_j\left(U_i \cap U_j\right) \to s_i\left(U_i \cap U_j\right) \tag{2}$$

Satisfying $\gamma_{ij} \circ s_j\big|u_i \cap U_j = s_i\big|u_i \cap U_j$. This implies that the level sets of the submersions S_i coincide on overlaps. The collection $\{\gamma_{ij}\}$ satisfies the cocycle condition $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$ and is called the Haefliger cocycle representing $\mathcal{F}$

(3) *Integrable Sub-bundle:* An integrable sub-bundle. F Of the tangent bundle TM of rank $\mathrm{n} - \mathrm{q}$. Integrability means that for any vector fields $\mathrm{X}, \mathrm{Y} \in \Gamma(\mathrm{F})$, their Lie bracket $[\mathrm{X}, \mathrm{Y}]$ also lies in $\Gamma(\mathrm{F})$ .We often denote $\Gamma(\mathrm{F})$ by $\mathcal{F}$ and $\mathcal{F}$ by $\mathrm{T}\mathcal{F}$. This condition ensures that the tangent spaces to the leaves form a smooth distribution.

(4) *Differential Ideal:* A locally trivial *differential ideal* $\mathcal{J} = \bigoplus_{\mathrm{k}=1}^{\mathrm{n}} \mathcal{J}^\mathrm{k}$ of rank q, in the de Rham complex $\Omega^\bullet(\mathrm{M})$. This ideal consists of differential forms that vanish when restricted to the leaves of the foliation.

When a manifold M is equipped with a foliation $\mathcal{F}$, the pair $(M, \mathcal{F})$ is called a *foliated manifold.*

The first two descriptions highlight the local structure of a foliation, emphasizing the decomposition into leaf directions and transverse directions. The third description captures the integrability condition crucial for the existence of leaves as immersed submanifolds. The fourth description connects foliations to the algebraic structure of differential forms.

The leaf space $M/\mathcal{F}$ is obtained by identifying points on the same leaf. Understanding the topology and geometry of the leaf space is a central theme in the study of foliations. However, $M/\mathcal{F}$ often lacks a good manifold structure, making its analysis challenging.

Since most foliations come from group action or symmetry, which tends to result in singular points, the study of only regular foliations is rather limited. While there are a few methods to work with singularities in conventional differential geometry, they are not universal. The use of higher categorical and homotopical theory, i.e., higher structures in. differential geometry, or so-called higher differential geometry in the recent progress of. Differential geometry is a promising way of solving singularity problems.

Let M be a smooth manifold and E be a smooth real vector bundle over M.

We use $C^\infty(M)$ to denote the algebra of smooth real valued functions on M .We use $C^\infty(M;E)$ to denote the module of smooth sections of the vector bundle E

Definition 2.2. A singular *foliation* on M is a submodule $\mathcal{F}$ of tangent module $T_M = C^\infty(M;TM)$ satisfying:

(1) $\mathcal{F}$ Is locally finitely generated.

(2) $\mathcal{F}$ Is involutive, i.e. $[\mathcal{F},\mathcal{F}] \subset \mathcal{F}$

Here are some simple examples:

Example 2.3. We consider a partition of $\mathbb{R}$ into 3 leaves: R, $\{0\}$ and $\mathbb{R}_+^*$. This partition corresponds to various foliations $\mathcal{F}_k$ with $k > 0$ where $\mathcal{F}_k$ is the module generated by the vector field $x^k \frac{\partial}{\partial x}$ and is different for each k

Example 2.4. We consider the partition of $\mathbb{R}$ into 2 leaves: $\mathbb{R}_+^*$ and $\{x\}$ for every $x \leq 0$. This foliation is defined by any module generated by the integral curves of any vector field $f\frac{\partial}{\partial x}$ where $f(x)$ vanishes for every $x \leq 0$.

Example 2.5. We consider the partition of $\mathbb{R}^2$ into 2 leaves: $\{0\}$ and $\mathbb{R}^2\backslash\{0\}$ given by the action of a Lie group G ,where G can be $\mathrm{GL}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{R})$,or $\mathbb{C}^*$ .While the foliation is different for each action, the corresponding $\mathcal{F}_x$ are equal to $T_x\mathbb{R}^2$ at each non-zero $x \in \mathbb{R}^2$ However, $\mathcal{F}_0$ is the Lie algebra $\mathfrak{g}$.

2.2. L_∞-algebroids.

Higher differential geometry uses higher categorical and homotopical methods to study higher structures in differential geometry and topology. Generally speaking, singularities in higher geometry imply that there exist hidden higher structures behind. We can then use homological aigebra or homotopical algebra to resolve it, which yields a new geoemtric objects. Among many new ideas in higher geometry, L_∞-algebroids are naturally associated with singular foliations. In [16] and [20] Laurent-Gengoux, Lavau, and T. Strobl discovered that for any singular foliation $\mathcal{F}$ which can be resolved by vector bundles, there exists a unique L_∞-algebroid g called the universal L_∞-algebroid naturally associated to $\mathcal{F}$. This L_∞-algebroid g can be thought as a homotopical replacement of $\mathcal{F}$. In fact, in [25] and [26], Nuiten developed a homotopy theory of L_∞-algebroid using semi-model categories. Under Nuiten's framework, we can then regard g as a cofibrant replacement of $\mathcal{F}$ in the category of L_∞-algebroid.

Roughly speaking, L_∞ -algebroid is a combination of generalizations of Lie algebra in two directions:

(1) (Horizontal categorification*)* Generalize the brackets of a Lie algebra to higher brackets, i.e., not only just 2-brackets, but also 3-brackets, 4-brackets etc. and these brackets satisfy a homotopical version of Jacobi identity, which means that the 2-bracket does not satisfy the (strict) Jacobiidentity, but satisfy Jacobi identity up to homotopy. This type of new algebraic objects is called L_∞*-algebras*

(2) Vertical categorification) If we regard Lie algebra as a category with one object,then we can generalize it to a category with many objects.We assume a set of objects to be a manifold M,where themorphism associated with it is a Lie algebra above each point,then globally we get a vector bundle A over M A naturally acts on M through the bracket. We call A a *Lie algebroid.*

Combining these two ways of categorification (vertically and horizontally) on Lie algebras, we can then get L_∞-algebroids.

Let us first define L_∞-algebras.

Definition 2.6. An L_∞-algebra is a graded vector space E equipped with a family of graded symmetric k -multilinear maps $(\{\cdots\}_k)_{k\geq 1}$ of degree +1, which holds true for the generalized Jacobi identities, i.e.

$$\sum_{i=1}^{n}\sum_{\sigma\in U_n(i,n-i)}\epsilon(\sigma)\left\{\{x_{\sigma(1)},\dots,x_{\sigma(i)}\}_i, x_{\sigma(i+1)},\dots,x_{\sigma(n)}\right\}_{n-i+1}=0 \tag{3}$$

For every homogeneous element $x_1,\dots,x_n\in E$, where $U_n(i,n-i)$ is the set of $(i,n-i)$ -unshuffles (the permutations σ of n elements which preserves the order of the first i elements and thelast $n-i$ elements:

$$\sigma(1)<\cdots<\sigma(i),\quad \sigma(i+1)<\cdots<\sigma(n) \tag{4}$$

And $\epsilon(\sigma)$ is the sign induced by the permutation of elements in the symm etric algebra of E.

Let us take note that these higher brackets actually correspond to cohomology of Lie algebra under deformation.

Next, let us take a look at another direction of categorification of Lie algebras.

Definition 2.7. A Lie algebroid over M is a vector bundle $A\to M$,equipped with a vector bundle morphism $\rho: A\to TM$, i.e., the anchor map, and a Lie bracket $[.,.]_A$ on $\Gamma(A)$ such that the Leibniz identity holds true:

$$\forall x,y\in\Gamma(A),\ f\in C^\infty(M)\quad [x,fy]_A=f[x,y]_A+\rho(x)[f]y, \tag{5}$$

And the Lie algebra homomorphism condition:

$$\forall x,y\in\Gamma(A)\quad \rho([x,y]_A)=[\rho(x),\rho(y)]. \tag{6}$$

Lie algebroid is one of the most important tools in studying the theory of foliation. We can regard Lie algebroids as the infinitesimal version of Lie groupoids. A Lie groupoid is a groupoid (a category where all morphisms are invertible) such that both objects and morphisms are smooth manifolds. For any foliations $\mathcal{F}$,there are several Lie groupoids that canbe constructed from $\mathcal{F}$, such as the *holonomy groupoid* $\mathrm{Hol}(\mathcal{F})$ and the *monodromy groupoid* $\mathrm{Mon}(\mathcal{F})$[22]. On the other hand, we can also construct (singular) foliations from Lie algebroids.In fact, given any Lie algebroid A,the image of its anchor map $\rho(A)\subset TM$ yields a singular foliation.

Definition 2.8. Let M be a smooth manifold. We denote the sheaf of functions of M by $\mathcal{C}$. Let E be a sequence $E=(E_{-i})_{i\leq 1\leq\infty}$ of vector bundles over M.A $L_\infty-algebroid$ (or$Lie\infty-algebroid$) structure on E is defined by:

(1) A degree 1 vector bundle morphismρ: $E_{-1}\to TM$ i.e. the anchor of the Lie ∞-algebroid.

(2) A family of graded symmetric k -multilinear maps $(\{\cdots\}_k)_{k\geq 1}$ of degree +1 on the sheaf of graded vector spaces $\Gamma(E)$ with the following constraints:

(1) Leibniz conditions:

(a)The unary bracket $d:=\{\cdot\}_1:\Gamma(E)\to\Gamma(E)$ is $\mathcal{O}$ -linear, i.e., it forms a family $d_i: E_{-i}\to E_{-i+1}$ of vector bundle morphisms, where $d_1=0$.

(b) For all $x\in\Gamma(E_{-1})$ and $y\in\Gamma(E)$, it satisfies

$$\{x,fy\}_2=f\{x,y\}_2+\rho(x)[f]y \tag{7}$$

Where $\{x,fy\}_2=f\{x,y\}_2$ for all $x\in\Gamma(E_{-i})$ with $i\geq 2$

(c)Each of the maps $\{\cdots\}_n$ is $\mathcal{C}$ -linear, for all $n \geq 3$
(2) Higher Jacobi identities
(a) $\rho \circ d_2 = 0$
(b) $d_{i-1} \circ d_i = 0$ For all $i \geq 3$
(c) $\rho(\{x, y\}) = [\rho(x), \rho(y)]$ For all $x, y \in \Gamma(E_{-1})$
(d)

$$\sum_{i=1}^{n} \sum_{\sigma \in U_n(i,n-i)} \epsilon(\sigma) \left\{\{x_{\sigma(1)}, \dots, x_{\sigma(i)}\}_i, x_{\sigma(i+1)}, \dots, x_{\sigma(n)}\right\}_{n-i+1} = 0 \tag{8}$$

For all $n \geq 2$, and for all homogeneous elements $x_1, \dots, x_n \in \Gamma(E)$

When $E_{-i} = 0$ for all $i \geq n+1$ a Lie ∞-algebroid structure over M is a Lie ∞-algebroid. In some literatures, for example [25] [26], Lie ∞-algebroids are referred to. L_∞-Algebroids as it's a globalization of L_∞-algebras.

Remark 3.9. Note that [20] and [16] use graded symmetric bracket $\{\cdots\}$ in the definition of L_∞ -algebroids. We will follow [25] [26] using the graded antisymmetric products $\lfloor\cdots\rfloor$ .As a consequence, [20] and [16] started the index -1 whereas we would start from 0.

One of the most important results in singular foliation theory is that people found the intrinsic relation between singular foliations and Lie ∞-algebroids. In fact, once a singular foliation admits a resolution by vector bundles (of finite amplitude), then we can construct a natural L_∞ -algebroids from the resolution.

Theorem I. *[20] [16] Let (E, Q) Bea universal Lie 00 -algebroid resolving a singular foli-. Ation $\mathcal{D}$ .Then,*

(1) For any Lie oo-algebroid (E', Q') that defines a sub-singular foliation of. $\mathcal{D}$ D $\mathcal{D}\big(i.e., \rho'\big(\Gamma(E'-1)\big) \subset D$), there is a Lie ∞ algebroid morphism from $(E', Q^{\bar{\prime}})$ to (E, Q) over the identity of. M And any two such lie ∞ -algebroid morphisms are homotopic.

(2) Two universal Lie 00 algebroids resolving the singular foliation D are isomorphic up to homotopy, and *two such isomorphisms are homotopic.*

2.3. Pontryagin classes.

The Pontryagin classes, named after the Russian mathematician. Lev Pontryagin, serve as a crucial topological invariant and offer important information about the geometric structures and topologies of vector bundles. By distinguishing non-isomorphic vector bundles, the Pontryagin classes are applied widely in different areas including the topology of manifolds and obstruction Theory. The Pontryagin algebra $\mathbb{Z}[p_1, \dots, p_{[n/2]}$ is defined using the cohomology of the classifying space of GL_n.

Definition 2.10. Let $E \to X$ be an n -dimensional vector bundle over a paracompact space X.Let

$$g_E : X \to BGL_n \tag{9}$$

Be the map, unique up to homotopy.

Theorem II. *[11]*

The polynomial ring $\mathbb{Z}[p_1, \dots, p_{[n/2]}(with\ \ p_i \in H^{4i}(BGL_n)$ canonically defined) is isomorphic to the cohomology ring $H^*(BGL_n)$: Therefore,

$$H^*(BGL_n; \mathbb{R}) \cong \mathbb{R}\big[p_1, \dots, p_{[n/2]}\big] \tag{10}$$

We define the i-th (real) Pontryagin class of E tobe

$$p_i(E) = g_E^*(p_i) \in H^{4i}(X; \mathbb{R}) \tag{11}$$

For $i = 1, \dots, \left\lfloor\frac{n}{2}\right\rfloor$.

Moreover, we define *the (real) Pontryagin ring* of. E Tobe the graded subring

$$\mathrm{Pont}^\bullet(E) := g_E^*\big(H^\bullet(BGL_n;\mathbb{R})\big) \subseteq H^\bullet(X;\mathbb{R}) \tag{12}$$

3. Cohesive modules on foliations

Cohesive modules, introduced by J. Block in [12], give systematic and convenient generalization of geometry objects such as derived categories of complex manifolds [12] and gerbes [10] and local systems on smooth manifolds of foliations [13]. This provides a useful tool in the analysis of differential and topological properties of foliated spaces by providing insights into how the local geometry of the leaves interacts with the overall geometric structure of the manifolds encoded by the $\mathbb{Z}$-connections

3.1. Cohesive modules over smooth manifolds.

Cohesive modules over smooth manifolds can be roughly regarded as complexes of vector bundles with $\mathbb{Z}$-connections. Module is a generalization of vector bundles. Particularly, the sections of the bundles are treated as modules over the ring of smooth functions on the manifold. In fact, by SerreSwan theorem, vector bundles over smooth manifolds are equivalent to finitely generated projective modules over the smooth functions. Moreover, the idea of cohesion ensures the compatibility with both the differential and geometric properties of the manifold using the Z -connections, which is a generalization of Quillen's superconnection in. [27] [23]

Definition 3.1. Let $A = (A^\bullet, d, c)$ be a curved dga. Note the A^0 usually corresponds to the "function" part of the dga, if we regard 0-5h degree as ordinary geometry and higher components as higher structures. For example, $\Omega^0(M) = C^\infty(M)$ is just the smooth functions on M .We define the dg *-category* $\mathcal{P}_A$ as follows?

(1) An object $E = (E_\bullet, \mathbb{E})$ in $\mathcal{P}_A$ is a cohesive module, which is a finitely generated and projective $\mathbb{Z}$ -graded (but bounded in both directions) right module $E_\bullet$ over A^0 together with a $\mathbb{Z}$-connection that

$$\mathbb{E}: E_\bullet \otimes_A A^\bullet \to E_\bullet \otimes_A A^\bullet \tag{13}$$

Satisfying the integrability condition such that the relative curvature vanishes

$$F_{\mathbf{E}}(e) = \mathbb{E} \circ \mathbb{E}(e) + e \cdot c = 0 \tag{14}$$

For all $e \in E_\bullet$.

(2) The morphisms of degree k. $\mathcal{P}_A^k(E_1, E_2)$ between two cohesive modules $E_1 = (E_{\bullet 1}, \mathbb{E}_1)$ and $E_2 = (E_{\bullet 2}, \mathbb{E}_2)$ of degree k are

$$\{\phi: E_{\bullet 1} \otimes_A A^\bullet \to E_{\bullet 2} \otimes_A A^\bullet \mid \text{Of degree } k \text{ and } \phi(ea) = \phi(e)a,\ \forall a \in A^\bullet\} \tag{15}$$

With the standard differential defined

$$d(\phi)(e) = \mathbb{E}_2\big(\phi(e)\big) - (-1)^{|\phi|}\phi\big(\mathbb{E}_1(e)\big) \tag{16}$$

Therefore,

$$\mathcal{P}_A^k(E_1, E_2) = \mathrm{Hom}_A^k(E_{\bullet 1}, E_{\bullet 2} \otimes_A A^\bullet) \tag{17}$$

Let $E = (E_{-i})_{i\geq 1}$ be a sequence of positively graded vector bundles over a manifold M. We then have a one-to-one correspondence between NQ-manifold structures on E and Lie ∞ -algebroid structures over E . Further, for any singular foliation $\mathcal{F}$ which can be resolved by vector bundles, the foliation dga associated to $\mathcal{F}$ be $A = \mathrm{Sym}\mathfrak{g}^\vee[-1]$ where g is the L_∞ -algebroid constructed from the resolution. Then we can consider P_A which is the dg category of cohesive modules over g , hence can be regarded as cohesive modules over the singular foliation $\mathcal{F}$.

One special case is to consider the anti-homomorphic tangent bundle as a complex foliation on $T_{\mathbf{C}}M$, then we have the following theorem which generalize derived categories on complex manifolds

Theorem 3.2 ([12]). *Let X be a compact complex manifold, and* $\mathfrak{g} = T^{0,1}X$ *be the Dolbeault Lie algebroid. The homotopy category of the dg-category* $Mod_{CE(g)}^{Coh} = Rep_{g,A}$ *is equivalent to the bounded derived category of chain complexes of sheaves of O x-modules with coherent cohomology on* X.

3.2. Pontryagin classes of cohesive modules.

One direct generalization of Pontryagin classes of the normal bundles associated to a foliation is to consider the Pontryagin classes are defined using a Lie pair, i.e., a Lie algebroid A and one of its Lie subalgebroids. Usually, Lie algebroids provides lots of examples of singular foliations, hence this approach is already an important tool in singular foliation theory. In this paper, in order to deal with a more general type of singular foliation, we aim to generalize this by using an L_∞ -pair, i.e., an L_∞ -algebroid and one of its L_∞ subalgebroids, to provide an alternative definition. Of Pontryagin classes.

Definition 3.3. Let us consider a L_∞ -algebroid $g \to M$and a vector bundle $E \to M$ of rank k with a linear g –connection

$$\nabla: \Gamma(A) \times \Gamma(E) \to \Gamma(E). \tag{18}$$

Suppose$R_\nabla \in \Omega^2(\mathfrak{g}, \operatorname{Hom}(TM, \mathfrak{g}))$ to be the curvature of ∇. We define the form $R_\nabla^i \in \Omega^{2i}(\mathfrak{g}, \operatorname{End}(E))$ for $i \geq 1$ by:

$$\widehat{R_\nabla^i} = \widehat{(R_\nabla)}^i \in \operatorname{End}_{\Omega^\bullet(\mathfrak{g})}\big(\Omega^\bullet(\mathfrak{g}, E)\big). \tag{19}$$

Then, by

$$\left[d_\nabla, \widehat{K}\right] := d_\nabla \circ \widehat{K} - (-1)^k \widehat{K} \circ d_\nabla = \widehat{d_{\nabla \operatorname{End} K}}. \tag{20}$$

We have:

$$\widehat{d_{\nabla \operatorname{End}} R_\nabla^i} = \left[d_\nabla, R_\nabla^i\right] = \left[d_\nabla, \widehat{(R_\nabla)}^i\right] = \left[d_\nabla, d_\nabla^{2i}\right] = 0 \tag{21}$$

Therefore, with

$$d_{A\left(\operatorname{fr}(R_\nabla^i)\right)} = \widehat{\operatorname{tr}}\left(d_{\nabla \operatorname{End}} R_\nabla^i\right) = 0. \tag{22}$$

$\widehat{\operatorname{tr}}\left(R_\nabla^i\right)$ Defines (a cohomology class) in $H^{2i}(\mathfrak{g})$

Definition 3.4. Let. E Be a vector bundle over M and let $g \to M$ Bea L_∞ -algebroid.

The g-Pontryagin *characters of.* E Are defined for anylinear g -connection ∇ on E .The cohomology classes $\sigma_{\mathfrak{g}}^i(E) := \left[\widehat{\operatorname{tr}}\left(R_\nabla^i\right)\right] \in H^{2i}(\mathfrak{g})$, for $i \geq 1$.

The g-Pontryagin *algebra of* E is the $\mathbb{R}$ -subalgebra $\operatorname{Pont}_{\mathfrak{g}}^\bullet(E) \subseteq H^\bullet(g)$ generated by the$g$ -Pontryagin characters.

Definition 3.5. The (characteristic polynomial) is defined as

$$\det(\lambda \cdot I_k + X) = \sum_{i=0}^{k} f_i(X)\lambda^{k-i} \tag{23}$$

For i a positive integer which is the homogeneous polynomials f_i of degree i on $\operatorname{gl}(k, \mathbb{R})$ for $k \geq i \geq 0$

Definition 3.6. Since these characteristic polynomials are obviously $\operatorname{Gl}(k, \mathbb{R})$ -invariant, for each $i \geq 1$ we define the i -th g -Pontryagin class of E as:

$$p_{\mathfrak{g}}^i(E) := \left[f_{2i}\left(\frac{1}{2\pi} R^\nabla\right)\right] \in H^{4i}(\mathfrak{g}) \tag{24}$$

For any choice of connection $\nabla : \Gamma(g) \times \Gamma(E) \to \Gamma(E)$ .The g -Pontryagin classes of E generate together $\operatorname{Pont}_{\mathfrak{g}}^\bullet(E)$. The total 9 -Pontryagin class of E is defined by

$$p_{\mathfrak{g}}(E) = \left[\det\left(I_k + \frac{1}{2\pi}R^{\nabla}\right)\right] = 1 + p_1^{\mathfrak{g}}(E) + p_2^{\mathfrak{g}}(E) + \cdots + p_{\left\lfloor\frac{k}{2}\right\rfloor}^{\mathfrak{g}}(E) \in \mathrm{Pont}_{\mathfrak{g}}^{\bullet}(\mathrm{E}) \tag{25}$$

3.3. Bott connection on singular foliations.

Bott connection offers a way of differentiation along the leaves of a foliation, which is significant in helping grasp the cohomological and homotopical properties of foliations via the characteristic classes.

Definition 3.7. For the $C^{\infty}(M)$ -module $\mathcal{N} = \mathcal{X}(M)/\hat{\mathcal{F}}$,the Lie bracket of vector fields descends to a map $\hat{\mathcal{F}} \times N \to N$ .After fixing a leaf L ,let us consider the Lie algebroid $A_L \to L$,whose sections are given by $\mathcal{F}_b/\mathcal{I}_L\mathcal{F}_b$ .Let us also consider the normal bundle $N_L = \frac{T_IM}{TL} \to L$, whose space of sections is $\mathcal{N}/\mathcal{I}_L\mathcal{N}$.

The *Bott* connection on N_L is induced by the Lie bracket as follows:

$$\nabla^{L,\perp}: C^{\infty}(L; A_L) \times C^{\infty}(L; N_L) \to C^{\infty}(L; N_L), \quad \nabla^{L,\perp}_{\langle X\rangle}\langle Y\rangle = \langle [X, Y]\rangle. \tag{26}$$

The map $\nabla_{L,\perp}$ is a flat Lie algebroid connection, i.e., a Lie algebroid representation of A_L on N_L .This thus can be considered the same as a Lie algebroid morphism

$$\nabla^{L,\perp}: A_L \to \mathrm{Der}(NL). \tag{27}$$

Where $\mathrm{Der}(NL)$ is the Lie algebroid over L whose sections are given by $\mathrm{CDO}(N_L)$ i.e., the first order differential operators $D: C^{\infty}(L; NL) \to C^{\infty}(L; NL)$ such that there exists a vector field $\sigma_D \in \mathfrak{X}(M)$ with $D(fX) = fD(X) + \sigma_D(X)(f)X$

4. Baum-bott theorem for smooth singular foliations

Baum and Bott made significant strides in the study of singularities in the context of holomorphic vector fields on complex manifolds [9]. They pioneered the study of residues of singularities of holomorphic foliations, a crucial aspect in understanding the. Underlying topology and geometry of the foliation. However, their work was primarily focused on the case where the dimension of the connected components of the singular set was equal to $(r-1)$, where r is the dimension of the leaves of the foliation.

Lazarov and Pasternack [21] extended Baum and Bott's work [3] and explored the residues of singularities of a Riemannian foliation defined on a Euclidean space.They considered a more specific setting, with only a single singularity at the origin.Their work contributed to our understanding of how singularities can shape the global properties of a foliation.

Furthering this line of research, S. Nishikawa [24] generalized Lazarov and Pasternack's findings to projective foliations, thereby expanding the scope of the study to a more general and abstract setting. However, like Baum and Bott, Nishikawa's study also imposed a restriction on the dimension of the singularset of the foliation.

Our work aims to continue this trajectory by considering general singular foliations on smooth manifolds and providing an explicit formula for the residues of the connected components of the singular set, without imposing any restrictions on its dimension.The crux of our approach lies in reducing the problem of computing the residues of the singular set of the foliation to the problem of computing the residues of a zero set of a vector fieldon M .To achieve this,we leverage the method of transgression,a powerful technique in differential geometry due to Chern and Weil.This approach allows us to bypass the limitations of previous studies and provides a more general framework for studying the residues of singularities in Riemannian foliations.

4.1. Bott Vanishing theorem.

Definition 4.1. A basic connection ∇ on NF is one such that

$$\nabla_X(Z) = \pi[X, \tilde{Z}], \tag{28}$$

For all $X \in \Gamma(E)$ where $\tilde{Z} \in \mathcal{X}$ is such that $\pi(\tilde{Z}) = Z$

Lemma 4.2. *Under the* assumption *that* E *is integrable, there exists a basic connection on NF.*

Lemma 4.3. *Let* ∇ Bea *basic connection on NF, and* k *the* curvature *of* ∇.*Then* $k(X, X') = 0$ *for all* $X, X' \in \Gamma(E)$.

Proof. Let $Z \in \Gamma(NF)$ and $\tilde{Z} \in \mathcal{X}$ with $\pi(\tilde{Z}) = \mathbb{Z}$ .Then

$$\begin{aligned} k(X, X')(Z) &= \nabla_X \nabla_{X'}(Z) - \nabla_{X'} \nabla_X(Z) - \nabla_{[X,X']}(Z) \\ &= \nabla_X(\pi[X', Z]) - \nabla_{X'}(\pi[X, \tilde{Z}]) - \pi\left[[X, X'], \tilde{Z}\right]. \end{aligned} \tag{29}$$

But we can choose

$$\begin{aligned} \pi[X, \tilde{Z}] &= [X, Z], \\ \pi[X', \tilde{Z}] &= [X', Z], \end{aligned} \tag{30}$$

SO

$$k(X, X')(Z) = \pi\left[X, [X', \tilde{Z}]\right] - \pi\left[X', [X, \tilde{Z}]\right] - \pi\left[[X, X'], \tilde{Z}\right] = \pi(0) = 0 \tag{31}$$

By the Jacobi identity.

Lemma 4.4. *Let* $U_\alpha \subset M$ *be a simultaneously trivializing neighborhood for NF and* E , σ_α *a smooth frame for NF over* U_α . *Let* $I_\alpha(E)$ *be the ideal in* $A^*(U_\alpha)$ *generated by those 1-forms which vanish on* $\Gamma(E|_{U_\alpha})$. *Let* k^u *be the curvature matrix associated to the frame* $U\alpha$ *by a basic connection. Then each* $k_{ij}^\alpha \in I_\alpha(E)$.

Proof. Over U_α E can be described as the set of tangent vectors on which certain 1-forms $\theta_1, \ldots, \theta_q$ vanish, these 1-forms being linearly independent at each point of U_α .In particular, $I_{\tilde{\alpha}}(E)$ is generated by $\theta_1, \ldots, \theta_q$ .Complete these to a basis of 1-forms by $\theta_{q+1}, \ldots, \theta_n$ these last restrict to a basis of $E_{p'}^*, \forall p \in U_\alpha$. Consider a nontrivial form

$$\omega = \sum_{q+1 \le i < j \le n} g_{ij}\, \theta_i \wedge \theta_j \tag{32}$$

Clearly, there are $X, X' \in \Gamma(E|_{U_\alpha})$ such that

$$\omega(X, X') \neq 0. \tag{33}$$

By Lemma (6.3), it follows that each $k_{ij}^\alpha \in I_\alpha(E)$.

Theorem I (Bott Vanishing Theorem [11]). *If* $E \subset TM$ *is* integrable *and if the quotient bundle* $NF = TM/E$ *has fiber dimension.* U *, then* $Pont^\bullet$ $(NF) = 0$ *for* $k > 2q$.

This is really a global integrability condition. Indeed, for any n-dimensional bundle, set

$$p(E) = 1 + p_1(E) + \cdots + p_{[n/2]}(E). \tag{34}$$

Because the leading term is 1, this is an invertible element of the ring $H^\bullet(X; \mathbb{R})$ of formal infinite series $a_0 + a_1 + \cdots + a_r + \cdots$, $a_i \in H^i(X; \mathbb{R})$ .If $E' \subset E$ is a subbundle, the basic "duality" formula holds:

$$p(E/E') = p(E')^{-1} p(E), \tag{35}$$

And shows that thePontryagin classes of E/E' depend only on theisomorphism classes of E and E' and not on the embedding $E' \subset E$. Thus, we can reformulate$(*)$ as follows.

Theorem II (Bott vanishing theorem version II [11]). *If* $E \subset TM$ *is a subbundle which is isomorphic to an integrable subbundle* $E' \subset TM$ *and if* $NF = TM/E$,$q = dim(NF)$ *then* $Pont^\bullet$ *(NF) = 0* $for\ k > 2q$

4.2. Bott Vanishing Theorem on Lie Algebroids.

A direct generalization of Bott Vanishing theorem is to generalize Pontryagin classes of the normal bundle to an involutive subbundle of the tangent bundle to the case of normal bundle (or representation).Since the core object is aflat F -basic connection on a smooth vector bundle TM/F , that can be extended to alinear TM -connection to define Pontryagin characters or classes, one can easily prove a similar result for the existence of a flat partial connection on a smooth vec tor bundle.Furthermore,the construction is adapted to the more general A -Pontryagin (where A is a Lie algebroid over M) classes of a vector bundle E.

Let A be a Lie algebroid over a smooth manifold M ,and let B be a subalgebroid of A over M .Let n be the rank of A ,and l be the rank of B .Set $q := n - l$, the rank of A/B i.e.,the codimension of B ,or the annihilator B° of B .Let E be a smoothvectorbundle over M ,with a flat B -connection ∇ .It is not difficult to see that ∇ can be extended to an A -connection $\bar{\nabla}: \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$, satisfying

$$\widetilde{\nabla}_{\mathrm{b}}\mathrm{e} = \nabla_{\mathrm{b}} \tag{36}$$

For all $b \in \Gamma(B)$ and $e \in \Gamma(E)$.

Define the space $I^\bullet(B) \subset \Omega^\bullet(A)$ as the ideal in $\Omega^\bullet(A)$ generated by the 1-forms vanishing on B .That is, it is generated by the sections of the annihilator $B^\circ \subset A^*$ of B. It is explicitly given by $I^0(B) = \{0\} \subset \Omega^0(A) = C^\infty(M)$ and

$$\mathrm{I^r(B)} = \omega \in \Omega^{\mathrm{r}}(\mathrm{A}) \mid \omega(\mathrm{b}_1, \ldots, \mathrm{b_r}) = 0 \text{ for all } \mathrm{b}_1, \ldots, \mathrm{b_r} \in \Gamma(\mathrm{B}) \tag{37}$$

For $r \geq 1$

Choose an open set $U \subset M$ trivializing A and B .That is,there is a smooth frame $\left(a_{1,\ldots,}a_n\right)$ for A over UI such that $\left(a_{q+1}, \ldots, a_n\right)$ is a smooth frame for B .Consider the dual frame $(\alpha_1, \ldots, \alpha_n)$ of A^* over U . By construction, $\left(\alpha_1, \ldots, \alpha_q\right)$ is a smooth frame for B° over U .Since $I^\bullet(B)$ isgenerated as an ideal by $\Gamma(B^\circ)$ for $r \geq 1$, an element UU of $I^r(U)$ canbe written as?

$$\omega = \sum_{i=1}^{q} \omega_i \wedge \alpha_i \tag{38}$$

With $\omega_i \in \Omega_U^{r-1}(A)$. Therefore, since B° has rank U , the wedge product

$$\left(I^\bullet(B)\right)^{q+1} = I^\bullet(B) \wedge \cdots \wedge I^\bullet(B) \quad \text{(q+1 times)} \tag{39}$$

Must necessarily vanish. It is now easy to see that

$$R_{\bar{\nabla}}(b, b')e = R_{\nabla}(b, b')e = 0 \tag{40}$$

For $b, b' \in \Gamma(B)$ and all $e \in \Gamma(E)$,and so $R_{\bar{\nabla}} \in I^2(B) \otimes_{C^\infty(M)} \Gamma(\mathrm{End}\,(E))$. This implies $R_{\bar{\nabla}}^i \in \left(I^2(B)\right)^i \otimes_{C^\infty(M)} \Gamma(\mathrm{End}(E))$. and so $R_{\bar{\nabla}}^i = 0$ for $i > q$ .More generally, for a $GL(k, \mathbb{R})$ invariant polynomial of degree d on $\mathfrak{g}[(k, \mathbb{R})$,the $2d$ -form $P(R_{\bar{\nabla}}) \in \Omega^{2d}(A)$ is an element of $\left(I^2(B)\right)^d$,and so $P(R_{\bar{\nabla}}) = 0$ for $d > q$

As a summary, this section has proved the following result.

Theorem III (Bott Vanishing Theorem for Lie algebroids [17]). *Let E be a smooth vector bundle over a smooth manifold M andlet A be a Lie algebroid over M .If there existsa Lie subalgebroid B of A of codimension q with a linear partial representation $\nabla: \Gamma(B) \times \Gamma(E) \rightarrow \Gamma(E)$, then the Pontryagin algebra*

$$\mathrm{Pont}_A^\bullet(E) \subset H^\bullet(A) \tag{41}$$

Are all trivial for $l > 2q$

Remark 5.5. *Given an ordinary linear connection* $\nabla: \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ *on a vector bundle* E *of rank* k, *a Lie algebroid* $A \to M$ *defines a linear* A *-connection* $\nabla^A: \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ *by*

$$\nabla_a^A e = \nabla_{\rho(a)} e. \tag{42}$$

It is easy to see that

$$[p(R_{\nabla^A})] = \rho^\star[p(R_\nabla)] \in H^\bullet(A) \tag{43}$$

For any $\mathrm{GL}(k, \mathbb{R})$ -invariant polynomial p on $gl(k, \mathbb{R})$ .Here, $\rho^\star$ is the cochain map

$$\rho^\star: (\Omega^\bullet(M), d) \to (\Omega^\bullet(A), d_A), \tag{44}$$

Defined by

$$\rho^\star(\omega)(a_1, \dots, a_s) = \omega(\rho(a_1), \dots, \rho(a_s)) \quad \text{for } \omega \in \Omega^s(M) \text{ a} \tag{45}$$

As observed by Fernandes in [15], these yields

$$\mathrm{Pont}_A^\bullet(E) = \rho^\star(\mathrm{Pont}^\bullet(E)) \tag{46}$$

Or more precisely

$$\mathrm{cw}_A(E) = \rho^\star \circ \mathrm{cw}(E). \tag{47}$$

Therefore, it is not hard to show the following obstruction result in terms of the classical Pontryagin algebra of E.

Corollary 3.2. Let E be a smooth vectorbundle over a smooth manifold, M and let A be a Lie algebroid over M . If there exists a Lie subalgebroid B of A of codimension q with a linear representation $\nabla: \Gamma(B) \times \Gamma(E) \to \Gamma(E)$, then the Pontryagin algebra

$$\mathrm{Pont}^\bullet(E) \subset H^\bullet(M) \tag{48}$$

All lie in the kernel of $\rho^*: H^\bullet(M) \to H^\bullet(A)$ for $l > 2q$

If a Lie algebroid A has a subalgebroid B of codimension q, then B is represented on A/B via the flat Bott connection

$$\nabla^B: \Gamma(B) \times \Gamma(A/B) \to \Gamma(A/B), \quad \nabla_b^B \bar{a} = [b, a]. \tag{49}$$

Hence, $\mathrm{Pont}_A^\bullet(A/B) \subset H^\bullet(A)$ is trivial for $l > 2q$. This yields obstructions to a subalgebroid structure on $B \subset A$ of codimension q.

However, in the case $A = TM$ and $B = F_M$ the algebroid FM is in fact more than just a subalgebroid, which implies the classical Bott Vanishing theorem.

In this section, we consider a singular foliation $\mathcal{F}$ which has a resolution by vector bundles, and $\mathcal{G} \subset \mathcal{F}$ being a sub-foliation. It's natural to ask whether the Bott vanishing theorem is valid in this setting

Proof.

4.3. Residue existence theorem for singular foliations.

Let $\mathcal{F}$ be a singular foliation which admits its resolution

$$0 \to \Gamma(E_{-n}) \to \Gamma(E_{-n+1}) \to \cdots \to \Gamma(E_{-1}) \to \Gamma(E_0) \to \mathcal{F} \to \tag{50}$$

And g is the universal L_∞ -algebroid over A associated with the complex of vector bundles. By assumption, F is a perfect complex over $C^\infty(M)$, this implies that the normal module of $\mathcal{F}$, i.e., $\mathcal{F}^\perp$ is also aperfect complex, which admits anatural resolution since

$$0 \to \Gamma(E_{-n}) \to \Gamma(E_{-n+1}) \to \cdots \to \Gamma(E_{-1}) \to \Gamma(E_0) \to TM \to \mathcal{F}^\perp \to 0, \tag{51}$$

Is exact. In other words, we define NF to be the complex

$$0 \to \Gamma(E_{-n}) \to \Gamma(E_{-n+1}) \to \cdots \to \Gamma(E_{-1}) \to \Gamma(E_0) \to TM \to 0, \tag{52}$$

Which is quasi-isomorphic to $\mathcal{F}^{\perp}$. Therefore, we can regard $\mathcal{F}$ is a subalgebroid of a tangent module T_M as and L_∞ -algebroid, i.e., $(T_M, \mathfrak{g})$ is an L_∞ -pair. Note that, we have the quasi-isomorphism $T_M \simeq E_\bullet \oplus NF_\bullet$ .Using the L_∞ -structure on $\mathcal{F}$we can build partial $\mathbb{Z}$ -connection over the foliation dga A.

Proposition 4.6. *[BZ] There exists a Bott* $\mathbb{Z}$ -connection $\mathbb{E}$ *over the foliation dga* A *associated to* $\mathcal{F}$*:*

$$\mathbb{E}: E_\bullet \otimes_A A^\bullet \to E_\bullet \otimes_A A^\bullet \tag{53}$$

Where $A = A^\bullet$ *is the foliation dga*

NF.Note that using Prop 16.3 in [29] and Lemma 6.19 in [25], this is equivalent to the case that $E \oplus NF$ carries an L_∞-structure, where the brackets vanish with more than one inputs from NF.

This Bott connection naturally lifts to a full $\mathbb{Z}$ over the de Rham dga $\Omega^\bullet(M)$,sincelocally $\mathcal{F}$ is a submersion, and then we can always lift a local section s of $\mathcal{F}$ to a local section $\tilde{S}$ of T_M with $\pi(\tilde{s}) = s$ .Equivalently, we want to lift the output of all brackets with more than one inputs from NF.

Lemma 4.7. *There exists a lift of E to a* $\mathbb{Z}$ *-connection* $\mathbb{E}$ *over the de Rham dga, i.e., we can lift the* L_∞*-algebroid structure on* $\mathbb{E}$ *to an* L_∞*-algebroid on* $\mathrm{E} \oplus NF$

Also notice that $E \oplus NF \simeq T_M$, where the quasi-equivalence is induced by the natural map

$$\begin{tikzcd} \Gamma(E_0) \arrow[r, "\rho"] & F \arrow[r, "\iota"] & T_M \\ T_M \arrow[ur, "\pi"] \arrow[urr, "\iota"'] & & \end{tikzcd} \tag{54}$$

Let $\mathbb{E}$ denote the Bott $\mathbb{Z}$ -connection. Now we have lifted the cohesive module NF with a Bott ZL -connection on over the foliation dga $A = \mathrm{Sym}\mathfrak{g}^\vee[-1]$ to a full $\mathbb{Z}$-connection over the de Rham dga $\Omega^\bullet$ to TM as $(TM, \widetilde{\mathbb{E}})$. We consider the characteristic classes computed from $\widetilde{\mathbb{E}}$.

Let $S = \cup S_i$ be a finite union of singular sets, where each $\overline{S_i}$ is compactly supported.

Theorem IV (Bott Vanishing Theorem for singular foliations). *The pontryagin classes of the normal module* $\mathcal{F}^\perp$ *,vanishes on* $M|$ $MM|_{M\setminus N}$ $M|_{M\setminus N} for\ Pont^l = 0$ *for* $l > 2q$, *= 0 =01 >2q* $l > 2q$. *where* $N = \cup_i N_i$ *and each* N_i *is an arbitrarily small open neighborhood of.* $S_{i,}$ *and* q *is the codimension of theregular leaves.*

Proof. Note that on $M\setminus N$, the anchor map μ has constant rank and $\mathcal{F}|_{M\setminus N}$ regular foliation, which implies that the linear part of g is quasi-isomorphic to a single term complex E_0, which is then isomorphic to the vector bundle underlying $\mathcal{F}$on $M\setminus N$. Now NF $|_{M\setminus N}$ is quasi-isomorphic to $\Gamma(E_0) \to T_M$, i.e. $E_i = 0, i \geq 0$ .As a result, all components of E vanish except

$$\Gamma(E_0) \to \Gamma(E_0) \otimes \Omega^1(M). \tag{55}$$

Now, by the Bott Vanishing Theorem for Lie algebroid (Theorem III), regarding $\Gamma(E_0)\,|_{M\setminus N}$ as a subalgebroid of $T_M\,|_{M\setminus N}$,$\mathrm{Pont}^l(NF)\,|_{M\setminus N} = 0$ for $l > 2q$.

Therefore, the Pontryagin class is contained in S , which means it is only non-zero in S the singular sets, which is the residue.

Definition 4.8. Let the singular sets be $S = \cup S_{i,}$ where each S_i is a connected component which is compact supported. The residue $\mathrm{Res}_p(\mathcal{F}, S_i)$ is defined as follows:

(1) A $\mathbb{Z}$ -connection IE on NF associated with $\mathcal{F}$ over M

(2) An open ε -neighborhood N of S_i whose closure is compacted supported, where εan arbitrarily small real number is.

Note that given a $\mathbb{Z}$ -connection, any closed differential form. $\alpha_i(\mathbb{E})$ which represents. $p_j(NF)$. is a globally well-defined form. The residue at S_i is thus defined as

$$\mathrm{Res}_p(\mathcal{F}, x) = \int_M \phi\left(\alpha_1(\mathbb{E}), \dots, \alpha_{k-1}(\mathbb{E})\right) \in \mathbb{R} \tag{56}$$

Where the polynomial $\phi(\alpha_1, \ldots, \alpha_{k-1})$ vanishes outside N.

Residue Existence Theorem for singular foliations. With the well-defined residue, we note that the construction pulls back to U' under the diffeomorphism $f: U' \to U$, and that changing the orientation of M barely changes the sign of the integral.

Let M be compact. Let $\mathbb{E}$ be a Bott $\mathbb{Z}$ -connection on NF over M .Let N_i be the **e** neighborhood for each $S_i \in S$ be W_i of such that N_i 's has pairwise disjoint closures. Let $\alpha_j(\mathbb{E})$ be a form representing $p^i(\mathbb{E})$ .Note that, from observation in Theorem IV, $\alpha_j(\mathbb{E})$ reduces to $\alpha_j(\nabla) \in \Omega^{4j}(M)$ on N_O where ∇ is the regular connection component of $\mathbb{E}$ where$N_O = M \backslash \cup_i N_i$.

From transgression formula in 4.3, we know that $\alpha_j(\mathbb{E})$ is closed, and is in $H^{4j}(NF \oplus \mathfrak{g})$ Since we have the quasi-isomorphism $\mathcal{F}^\perp \sim NF = NF \to T_M$ and $T_M \simeq NF \oplus \mathfrak{g}$ where $g \cong \mathcal{F}.\alpha_j(\mathbb{E})$ Is equivalent to a form in $H^{4j}(T_M) = H^{4j}(M)$ i.e. $\alpha_j(\mathbb{E})$ represents the Pontryagin class $p_j(M) \in H^{4j}(M)$.Therefore, we can finally complete the proof of the Residue Existence Theorem for Singular Foliations:

$$\begin{aligned}\sum_{S_i \in S} \operatorname{Res}_p(\mathcal{F}, S_i) &= \sum_{S_i \in S} \int_{N_i} \phi\left(\alpha_1(\mathbb{E}), \ldots, \alpha_{k-1}(\mathbb{E})\right) \\ &= \int_M \phi\left(\alpha_1(\mathbb{E}), \ldots, \alpha_{k-1}(\mathbb{E})\right) \\ &= \langle [\phi(\alpha_1(\mathbb{E}), \ldots, \alpha_{k-1}(\mathbb{E}))], [M] \rangle \\ &= \langle \phi(\alpha_1(\mathbb{E}), \ldots, \alpha_{k-1}(\mathbb{E})), [M] \rangle \end{aligned} \tag{57}$$

5. Conclusion

In conclusion, we extend the classical Bott Vanishing Theorem using the higher categorical and homotopical tools including L_∞-algebroids and cohesive modules to explore the Pontryagin classes on singular foliations. Our key findings include the demonstration of the conditions of vanishing and the applications of the residue theorem to foliations with singularities, which helps pave the way for future study of the characteristic classes on more general types of singular foliations on smooth manifolds. This is also an effort to translate the methods and toolsfrom algebraic geometry and complexgeometry to C^∞ geometry, which is intrinsically much more difficult than the previous two. In the future, wecould also investigate an infinitesimal formula to calculate the residue and then generalize it to the case of singularities with even fewer restrictions.

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