Introduction of Group Action and Examples of the Action

Yuanzhe Liu *
Oaks Christian School, Westlake Village, 91362, United States
* Corresponding Author Email: 20070507lyz@gmail.com

Abstract. The purpose of the essay is to provide a comprehensive understanding of key concepts in group theory, specifically focusing on group actions, orbits, and stabilizers. These important elements play a vital part in studying of algebraic structures and their symmetries. The author will begin by defining these concepts, explaining their significance, and illustrating them with practical examples. The discussion will include detailed definitions and explanations to build a solid foundation. In addition to these core concepts, the essay will introduce Sylow's Theorems and Burnside's Lemma. The author will explore how Sylow's Theorems helps the number and structure of subgroups within a finite group, and how Burnside's Lemma is used to count distinct objects under group actions. Through examples and detailed discussions, the author will demonstrate the practical applications of these theorems. Finally, readers are going to have a clear comprehension of how these mathematical tools are applied to solve problems and analyze group structures effectively. This works underscores the importance of group action in modern science.

Keywords: Group actions, Orbits and stabilizer; Group theory; Sylow's theorem.

1. Introduction

Group action is a fundamental concept in mathematics that illustrates how a group can interact with a specific set. This interaction provides a structured approach to examining symmetries and transformations. Analyzing group actions offers deep insights into the manipulation and comprehension of mathematical structures. In algebra, for example, group actions are crucial for exploring field extensions and polynomial equations. Particularly in Galois Theory, the Galois group's operation on a polynomial's roots is significant. It helps determine whether a polynomial can be solved using radicals, which is essential for solving such equations [1].

One main aspect of group actions is an element's orbit. The orbit includes element gotten by applying the group's action to a particular element. This results in a partitioning of the set into distinct classes, each corresponding to the group action. Another related concept is the stabilizer of an element, defined as the set of group elements that won't alter the element when acting on it [2]. The Orbit-Stabilizer Theorem plays a crucial role by connecting orbits and stabilizers. It establishes a connection in the size of a group, the stabilizer, and the orbit. This theorem is instrumental in understanding the underlying structure of groups and their actions [3].

Moreover, Burnside's Lemma will be introduced as a tool for tackling combinatorial problems. This lemma is useful to calculate the number of distinct ways to arrange beads on a necklace, considering various colors and symmetries. Additionally, the author will examine practical examples of group actions, such as the action of the D group on the vertices of an n-gon, the real numbers under multiplication acting on linear functions, and the cyclic group Z/n acts on the set of natural numbers. These examples emphasize the extensive applicability and importance of group actions in different mathematical scenarios.

2. Methods and Theory

2.1. Basic definitions

Let Ω be a set and gives a group G. A group action is a function $\rho: G \times \Omega \to \Omega$ satisfying $\rho(1_G, k) = k$ for every $k \in \Omega$ and m belonging to G, and $\rho(mm', k) = \rho(m, \rho(m', k))$ Denoting the action by $\rho(m, k) = m.k$, then

(Axiom1) $1_G . k = k$ for every k $\in \Omega$.

(Axiom2) $(mm').k = m.(m'.k)$ for every $m, m' \in G$ and $k \in \Omega$ [4]

The meaning is axiom 1 is that trivial element acts trivially, since every element x is sent back to itself when applying the identity element to the action. The meaning of axiom 2 is that g following by h is the same as hg.

There are some examples of group actions.

Example 1. The group Z/n with adding modulo n acts on natural number set under adding modulo n, denoted H. Define $Z/n \times H \to H$ that for $k \in Z/n, h \in H$ $(k, h) = k + h \bmod n$

Checking Axiom 1, $(0, h) = h$ acts trivially. Checking Axiom 2, for $j, k \in Z/n$, $j.(k.h)$=$j.(k + h \, mod \, n)$=$j + k + h \bmod n = (j + k) + h \, mod \, n = (jk).h$.

Example 2. The group of positive real numbers with the rule of under multiplication, denoted group G, do an action on set of all linear function with the form of $y = mx$,m>0, denoted Y. Define $R \times Y \to Y$ that for $p \in G$, $y \in Y$, $(p, y) = py$. The Axiom 1 is that $(1, y) = 1y = y$, acts trivially, while the Axiom 2 states that for $p, q \in G, p.(q.y) = p.(qy) = pqy = (pq)y = (pq).y$

Example 3. The dihedral group acts on the vertices. Define set of vertices $V = \{v1, v2 \ldots \ldots vn\}$ and $\varphi: D_n \times X \to X$. The Axiom 1 means that for $v \in V$, v=v acts trivially for the identity element in dihedral group. For Axiom 2, for $p, q \in D_n$, $p.(q.v) = (pq).v$.The explanation for axiom two is that the element in the dihedral group is either rotation or reflection, so applying them together is the same as applying separately.

Example 4. Let S = G and let G act by conjugation. For h $\in$ G and k $\in$ S, the group G acts upon S by conjugation by $(h, k) \to hkh - 1$. One can see that conjugation satisfies the two requirements for a group action. The Axiom 1 means that the group action is associative. For h, j $\in$ G and k $\in$ S, $h.(j.\mathrm{k}) = h(j.k)h^{-1} = h.(j.k.j^{-1}) \cdot h^{-1} = (h.j).k \cdot (h.j)^{-1} = (h.j).k$. in parallel, the Axiom 2 indicates that the action of the identity of G on k gives k [5].

Definition (orbit). Given a group G acting on a set X, the orbit of an element x∈X under the action of G is defined as $Orbit(x) = \{g \cdot x \mid g \in G\}$, where $g \cdot x$ denotes an element g acts on x.

An "orbit" refers to the set of all things in a set that can be obtained from a certain element under the action of a group. This set includes all possible results of applying every element. Orbits partition the set into disjoint subsets, each representing a unique method in which X can be acted by a group.

Definition (stabilizer). The set X is acted by Group G, the stabilizer of an element $x \in X$ is defined as $Stabilizer_G(x) = \{g \in G \mid g \cdot x = x\}$. The stabilizer of an element is a fundamental concept that describes how a group action "stabilises" or "fixes" certain elements of a set.

An example of relationship between orbit and stabilizer is that let $X = \{1,2,3,4\}$ and $G = S_4$ the element 1 is sent to 1,2,3,4. Then, the orbit of element 1 is $\{1,2,3,4\}$, which has a cardinality of 4. The stabilizer of 1 is all permutation which put 1 fixed, so it can be all permutation of 2, 3, and 4. So the stabilizer can be $\{id, (234), (432), (23), (34), (24)\}$, which has cardinality of 6. The author has $|S_4| = 24 = 4 \times 6 = |orbit \; of \; 1| \times |stabilizer \; of \; 1|$.

2.2. Basic Theorems

Orbit-Stabilizer Theorem. If G acts on a set H, $k \in K$ and $O_G\{k\}$is the orbit of k, then $o(O_G\{k\}) = [G: stabG(k)]$ [6].

Proof. $stabG(k)$Is a subgroup of G. define a map φ from $O_G\{k\}$ to the set of right cosets of $stab_G(k)$ in G by $(k.g)\varphi = (stab_G(k))g$, where $g \in G$.To show that this map is bijective? It is necessary to prove this is defined completely Suppose $k.g =.k$. Then $k.(gh^{-1}) = (k.g).h^{-1} = (k.h).h^{-1} = k$. That is, $kh^{-1} \in stabG(k)$ Hence $(stab_G(k))g = (stab_G(k))h$, as required.

Second, the author will show that this map is injective. Suppose $(stab_G(k))g = (stab_G(k))h$, and so $gh^{-1} = stab_G(k)$. Hence $k.h = (k.gh^{-1}).h = k.(gh^{-1}h) = k.g$. That is, φ is injective. Last, one must show that φ is surjective. For any $\left(stab_G(k)\right)g$, $g \in G$ then there is an element

$k.g \in O_G\{k\}$such that$(k.g)\varphi = (stab_G(k)g)$. So φ is surjective. Alternate expression: Let G acts on K, and let $k \in K$. The Orbit-Stabilizer Theorem states that $o(G) = |O_G\{k\}| \cdot |stab_G k|$.

Base on group action, there is also an important lemma, Burnside's Lemma. Let G be a group acting on a set S. The burnside Lemma states that: for a finite group G that acts on a finite set S, the author has $|S/G| = \frac{1}{G}\sum_{g\in G}|fix(g)|$. This means that $\sum_{g\in G}|fix(g)| = \sum_{g\in G}|\{s \in S: gs = s\}| = |\{(g,s) : g \in G, s \in S, gs = s\}| = \sum_{s\in S}|\{g \in G : gs = s\}|$. Since the stabilizer of s is $\{g \in G : gs = s\}$, one thus has $\sum_{g\in G}|fix(g)| = \sum_{s\in S}|stab(s)|$. By $\sum_{s\in S}\frac{1}{orb(s)} = \left|\frac{S}{G}\right|$ and with the orbit-stabilizer theorem, and one also has

$$\sum_{s\in S}\frac{1}{orb(s)} = \sum_{s\in S}\left|\frac{stab(s)}{G}\right| = \frac{1}{G}\sum_{s\in S}|stab(s)| \tag{1}$$

So $\left|\frac{S}{G}\right| = \frac{1}{G}\sum_{s\in S g\in G}|stab(s)fix(g)|$ is proved [7].

Sylow p-Subgroups. G is finite and with order $|G| = pk \cdot m$, p is a prime number. Then there exists a subgroup of G of order p^k, it is called a sylow p-subgroup. Arbitrary Sylow p-subgroups are conjugate. The number n_p of Sylow subgroup gives: $n_p \equiv 1(mod p)$ and n_p divides m.

To prove if a Sylow p-subgroup exist, the author uses the fact that the group G has order$p^k \cdot m$, where $p^k$is the most degree of p dividing the order of G. By the Sylow theorems, there must be one sub of G whose order is p^k. Two Sylow p-subgroups, P and Q, are conjugate. This means that for any Sylow p-subgroup P, every other Sylow p-subgroup can be obtained by conjugating P with some element of G. This property helps in understanding the symmetry and structure within the group. The condition that n_p divides m arises because the total number of Sylow p-subgroups, n_p must be a divisor of the index of any Sylow p-subgroup in G, which is m. The requirement that $n_p \equiv 1(mod p)$ stems from the way Sylow p-subgroups partition the group G, a result that is aligned with the Orbit-Stabilizer Theorem.

3. Results and Application

3.1. Applications of orbit-and-stabilizer theorem

The first is Symmetries of Polygons. Define Group Action such that for the dihedral group D_n, which consists of rotations and reflections, acts on the vertices of the polygon. The orbits is the distinct arrangements of vertices under the action of D_n. Vertexes stabilizer under D_n includes rotations and reflections that keep the vertices in place. One can use orbit-stabilizer Theorem to determine the number of distinct symmetrical arrangements or configurations [8].

The second is counting colorings. Count the ways to color a n-gon with k colors, two colorings are considered the same if one can be rotated to get the other. Define the group action which the symmetric group C_n acts on the set of colorings. The orbit each corresponds to a distinct pattern of colorings within rotation. The stabilizer of a coloring is determined by the symmetries of a certain coloring pattern. Applying the Orbit-Stabilizer Theorem can help to count the number of distinct colorings.

The third is Roots of Unity. Let ζ be an n-th root of unity. The group $C_n$of n-th roots of unity acts on itself by multiplication. The orbit of ζ under this action corresponds to all n-th roots of unity. Define the group action that: the cyclic group C_n acts under multiplication on the set of n-root of unity. The orbit of any n-root of unity is the entire set of n-root of unity, since applying the group action just send one root to another. The stabilizer of ζ is the subgroup of C_n that fixes ζ, which is usually trivial. Applying the theorem can determine the number of distinct sets of roots and their symmetries.

The fourth is Graph Automorphism. Consider a graph G and its automorphism group $Aut(G)$, which acts on the vertices of the graph. Define the group action that is the automorphisms of G act on its vertices. The orbit of a vertex are vertices that are equivalent in terms of graph structure. Vertex's stabilizer is the set of automorphisms that keep the vertex and its adjacent structure still. Using the Orbit-Stabilizer Theorem, one can determine the number of distinct graphs or in other word, in counting the number of automorphisms.

3.2. Application of Burnside's Lemma

Solving number of beads made with various color and bead's number. Suppose C is the set of colors that is used for the necklaces. Let C be a set. A necklace with C color is a sequence $x_1 \ldots x_n$ of elements. Two necklace is the same if they are rotation or reflection of each other. Given a necklace of 6 colors, with 6 color and n beads. The symmetric group of a 6-bead necklace isD_6. The order is 12 for D_6.

To use burnside's lemma, one needs to get $|fix(g)|$, for a necklace to be fixed under rotation 60°, the following bead has identical color as the previous bead, So, all of the beads on the necklace must have the same color, so $\mathrm{fix}(rotation\, 60°) = n$. Similarly one has $fix(e) = n^6$, $fix(rotation\, 120°) = 2n^2$ $fix(rotation\, 180°) = n^3$ $fix(rotation\, 120°) = 2n^2$ and $fix\big(reflection\ (3)\big) = 3n^4 + 3n^3$. Then, by the lemma one has

$$Totalnumber = \frac{1}{G}\sum_{g\in G}|fix(g)| = \frac{1}{12}(n^6 + 3n^4 + 4n^3 + 2n^2 + 2n). \tag{2}$$

4. Conclusion

A group action is a concept used to describe how a group can operate on a particular set. It offers a structured approach to studying symmetries and transformations of objects. Group actions are crucial for analyzing field extensions and polynomial equations. For example, the roots of polynomial which is acted by Galois group can reveal whether the polynomial can be obtained by radicals. The author will present several examples involving group actions. The element orbit includes all elements that is got by applying the action to that element. Orbits partition the set into distinct classes based on the group action. The stabilizer of an element denotes the subset of the group that leaves the element unchanged when acting on it. The orbit-stabilizer Theorem establishes a relationship among the sizes of the group, the stabilizer, and the orbit. Additionally, the author will introduce Burnside's Lemma, which is useful for tackling combinatorial problems, and provide examples to demonstrate its application. In the future, more study will be taken to learn the Cauchy's theorem and following essays about it will be written. Also, it will be necessary to provide a few examples or application of the theorem to stabilize the understanding.

References

[1] Jeremy, Gray. A History of Abstract Algebra: From Algebraic Equations to Modern Algebra. 1st edition, Springer Berlin Heidelberg, 2018.

[2] Pyone, A. Orbit-Stabilizer Theorem and Consequences. J. Myanmar Acad. Arts Sci, 2018, 16(3), 1-19.

[3] Feng, Y., Song, Z., & Xu, B. Cauchy Theorem, Sylow Theorems, and Orbit-Stabilizer Theorem in Group Theory. Highlights in Science Engineering and Technology, 2023, 47: 58–62.

[4] Dekker, J. C. E. Isols and Burnside's Lemma. Annals of Pure and Applied Logic, 1986, 32: 245–63.

[5] Klass, Michael J. A Generalization of Burnside's Combinatorial Lemma. Journal of Combinatorial Theory, Series A, 1976, 20(3): 273–78.

[6] Cohen, A., & Riche, S. Orbital Categories and Their Application to Algebraic Geometry. Journal of Algebraic Geometry, 2020, 29(3): 515-542.

[7] Burness, T. C., & Harper, S. (2020). Orbital Structure and Stabilizers in Displacement Groups. Mathematical Proceedings of the Cambridge Philosophical Society, 2020,169(1), 45-72.

[8] Isaacs, I. M. Finite Group Theory. American Mathematical Society Bulletin, 2017,125(4), 431-459.