

Theory and Applications of Orbits and Stabilizer in Group Action

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Abstract. The algebraic system of groups has become the focus of modern algebra courses because it is an important tool for studying symmetry problems. However, it is difficult for students to understand groups because it is difficult for them to understand how it can be used as a tool to study symmetry problems. In this article, the author presents a series of proofs, importance, and examples of the definition of orbits and stabilizer, the proof of useful claim, and the definition of orbits and stabilizer as an important part of swarm action. The characteristics of its derivative theorem orbits-stabilizers theorem are analyzed. Besides, the article also includes the application of Orbits-Stabilizer theorem to the Cauchy's Theorem, Lagrange's Theorem and Burnside's Lemma theorem by giving several definition, examples, and proof. Finally, this paper not only reviews the previous understanding and summary of the role of groups, but also contains the author's prospects for orbits and stabilizers in other fields.

Keywords: Group action; orbits; stabilizer; orbits-stabilizers theorem.

1. Introduction

In mathematical group theory, group action is a crucial concept, through which the symmetry of groups and their relationship with sets can be revealed. In this research area, orbits and stabilizer are the core concepts that provide an effective way to understand group structure. An orbit describes the trajectories of elements in a set under the action of a group, while a stabilizer depicts a group of elements that keep a particular element unchanged. Orbits and stabilizers not only occupy an important place in the study of pure mathematics but are also widely used as tools in other fields of mathematics, such as algebraic geometry and number theory.

In recent years, with the deepening of research, the application of ORBITS and stabilizer in many emerging fields has been further developed. For example, orbits-stabilizers theorem is used in combinatorial mathematics to calculate the number of invariant objects, which is especially important for solving symmetry problems. In addition, the application of orbits and stabilizers in representation theory, coding theory, and mathematical physics has also shown remarkable results. Besides, except that the direction in future is also various. One of them is on extending the concept of orbits and stabilizers to advanced algebraic structures field. This direction can lead to deeper insights into the classification of symmetries in algebraic varieties and geometric objects. Such that is an example focuses on nilpotent orbits in Lie algebras and their connection to representation theory and stabilizers [1].

The latest research literature further demonstrates the importance of ORBITS and Stabilizer in group action. For example, Cohen and Riche explore orbital categories and their application to algebraic geometry in their paper [2]. Guralnick and Kantor investigated the problem of probabilistic generation of finite mono-groups, in which the analysis of orbits and stabilizer played a key role in understanding the distribution of generators [3]. Burness and Harper's study focused on the orbital structure of the original displacement group and delved into the nature and application of stabilizer in this context [4].

Thus, it can be seen that the concepts of orbits and stabilizer not only deepen the understanding of group structure, but also provide powerful tools for problems in other areas of mathematics. These research results demonstrate the continued importance and wide application prospects of ORBITS and stabilizer in mathematical group theory.

2. Definition of orbit and stabilizer

2.1. Definition

Definition 1 (Orbit). The orbit of a specific element, when a group acts on it, consists of all possible elements that result from the group's action to s , formally expressed as $G \cdot s = \{g \cdot s | g \in G\}$ [5].

Definition 2 (Stabilizer). The set of elements in the group that leave the specific element unchanged under the group action s under a group action G is the set of group elements that leave s unchanged under the action, expressed as $Stab_G(s) = \{g \in G | g \cdot s = s\}$ [6].

Definition 3 (orbits- stabilizers theorem). Let G represent a collection of elements and a set, if for every element g within, associated with a permutation π_g on the set S is produced, one can say G exerts an action on S . The conditions are that the identity of G is required to a permutation on S that leaves every element of S unchanged, functioning as the identity, and when two elements g and h are combined, the resulting permutation $\pi(gh)$ of S will be $\pi_g \circ \pi_h$ [7].

2.2. Properties about Orbits and Stabilizer and its importance

Here the author discusses the function and nature of orbits from three aspects.

From set of Images to deduce the orbit $G(x)$ consists of all possible outcomes when different elements of G are applied to x . Beside the orbit can be used to establishes a relation that satisfies the properties of reflexivity, symmetry and transitivity. Over the set X . Two elements $y, x \in X$ are considered to be equivalent if there exists some $g \in G$ such that $g(x) = y$. This equivalence relation divides the set X into distinct, non-overlapping subsets. That show the equivalence relation characteristic. Partition of the Set: In an addition, the set X can be partitioned into disjoint orbits. Every element within X is contained in a single orbit, indicating that orbits are mutually exclusive and collectively exhaustive. That is the feature of Partition of the Set. The constant in subgroup characteristic, connection to symmetry role, in the Orbit-Stabilizer, theorem invariant under conjugation that four aspects. That is about Properties about stabilizer.

The stabilizer is a subgroup of the original group, adhering to the closure property when considering the operation of the group and containing the group's identity element. Beside the elements in the stabilizer represent symmetries of a specific point under the group action, meaning they leave the point unchanged. On the other hand the stabilizer's size is inversely related to the size of the orbit of the point, as described in the theorem that relates the orbit and stabilizer of an element. Finally, a stabilizer of a point remains invariant under conjugation within the group.

Combined with the above, it can be concluded that Orbit and stabilizer has the following functions as a mathematical tool. Orbits classify elements of X into disjoint subsets known as orbit classes. They reveal the structure and symmetry of X under G 's action. Stabilizers reflect the internal structure of properties of G and are useful for studying subgroups and so, orbits and stabilizers are fundamental tools in group theory and group actions. It provides a structured way to analyze symmetries across diverse, mathematical disciplines.

2.3. Proof two claims

Then, there are two useful claims- $G_x \leq G$ and the relation $x \sim g(x) \forall g \in G$ this relation fulfills the criteria for being classified as an equivalence relation.

Claim 1: $G_x \leq G$

To show that the stabilizer G is a group, the author needs to verify several group properties.

Proof: it is clear that $G_x \subseteq G$, So WTS G_x is a group. Firstly, from closeness aspect to deduce: if $h, h' \in G_x$, $hh'(x) = h(x) = x$ so hh' fixes x then $hh' \in G$. Secondly, from associativity aspect to see: as G is a group of symmetries (which is itself a group), this means for any $h, h', h'' \in G$ then $(hh')h'' = h(h'h'')$, the elements in G_x , must be associative. Finally, from the last aspect-

Inverse to see. Since G_x contains all elements that send x to itself, if $h \in G_x$ so is $\frac{1}{h} \in G$ because $\frac{1}{h}$ fixes x then $\frac{1}{h}(x) = x$ and hence $(h^{-1})^{-1} = h$ is in G .

Recall that the way in which group G operates on a particular set S defines a structure-preserving map between two algebraic structures of G to $Sym(X)$ and e in G so is a one that fixes x (i.e., $e(x) = x$). Since $e \in G$ must correspond to $id \in Sym(X)$ (the larger symmetry group) That is, e must be an element in G .

Claim 2: The relation $x \sim g(x) \forall g \in G$ is mapping satisfies the properties necessary to be considered an equivalence relation.

Proof. A relation is an equivalence, if there exists some $g \in G$ for which $y = g(x)$ satisfies the equivalence properties, the relation $x \sim y$ defined by $x \sim y$. The breakdown show from three aspects.

For any element $x \in X$, one needs to show $x \sim x$ since the identity element $e \in G$ satisfies $x = e(x)$ it follows that $x \sim x$, proving reflexivity. If there exists $g \in G$ then $x \sim y, y = g(x)$ then set $t = g^{-1}$ and the g^{-1} is also in G $t(y) = t(g(x)) = t \circ g(x) = x$. Since $t \in G, x = h(y), y \sim x$. If $x \sim y \sim z, y = g(x)$ and $z = g'(y) = g'(g(x)) = g'g(x)$. As $g'g \in G, x \sim z$, proving transitivity.

It also can further conclude that $G(x)$ is an equivalence class of x , and X is partitioned by the orbits of its elements and $\sim$ is an equivalence relation this implies that each element in Y belongs to an equivalence class represented by its orbit $G(x)$. The set X is partitioned into disjoint.

2.4. An Example

Consider $X = \{1, 2, 3\}$ and the symmetric group $G = S_3$. Consider the element $x = 1$ in X , $G = S_3$ can permute 1 to 1, 2 and 3, so the orbit $G(1) = \{1, 2, 3\}$, the order 3. Notice that $|G(1)| = 3$. The stabilizer of 1 under S_3 consists of all permutations in S_3 that leave 1 fixed. It can either swap elements 2 and 3 or leave 2 and 3 fixed. So $G_1 = \{id, (2\ 3)\}$ and it is a subgroup of S_3 with a cardinality of 2. Notice that $|S_3| = 6 = 2 \times 3$ because $G_1 = 2$ $|G(1)| = 3$ $|G_1| \times |G(1)| = 6$ so it can get the formula $|S_3| = |G_1| \times |G(1)|$. Connects the order of links the group with the order of the stabilizer and the size of the orbit.

3. Relation to Other Theorems in Group Theory

There are three parts to find relationship between serial theorems and orbits-stabilizer theorem, definition, relationship and example.

3.1. Definition

Definition (Cauchy's Theorem). In a finite group G , if its order is divisible by a prime p , there exists an element in G with order precisely equal to p [8].

Definition (Lagrange's Theorem). In every finite group G , the order of each subgroup H must be a number that evenly divides the group's total size of G . [9]

Definition (Burnside's Lemma). The total number of distinct orbits produced by the action of a finite group S on a set Y equals the average number of elements in Y that remain fixed under the group action (Number of Orbits = $\frac{1}{|S|} \sum_{s \in S} |Y^s|$) [10].

So, Cauchy's Theorem asserts that within any finite group G , when the order of the group is divisible by a prime number p , then there exists an element in G whose order is exactly p . According to Lagrange's Theorem, in a finite group G , the size of any subgroup H is a divisor of the size of G . Burnside's Lemma determines the count of unique orbits formed the operation of a finite group on a finite set X , by computing the mean number of fixed points for any element of the group.

3.2. Relationship between the orbit-stabilizer theorem and three theorems

Consider G acting on itself by conjugation. For any element $g \in G$, the orbit of an element as a result of this action is the set of all conjugates of g . The stabilizer in G corresponds to centralizer $C_G(g)$ of g . According to the theorem that relates the size of the orbit and stabilizer of an element under a group action, the size of the orbit is $[G : C_G(g)]$. By Cauchy's theorem, if p is a divisor of $|G|$, the group includes an element whose order equals p , there is an element g whose order is p . This element generates a cyclic subgroup of order p , and the corresponding stabilizer $C_G(g)$ has an index divisible by p , implying the existence of a non-trivial orbit, and hence a non-trivial conjugacy class.

If G acts transitively on a set X , then the Orbit-Stabilizer Theorem implies that the size of any orbit equals the index of the stabilizer subgroup. Since the size of the orbit divides $|G|$, Lagrange's theorem is satisfied. Specifically, the orbit size $|G : G_x|$ is the product of the stabilizer size $|G_x|$ and the number of distinct cosets. By summing up the contributions from the fixed points of each group element, Burnside's lemma is a method for calculating the number of distinct orbits in a group action.

3.3. Examples and proof the orbit-stabilizer theorem satisfies in each theorem:

3.3.1. Proof Cauchy's Theorem

Cauchy's Theorem example: Suppose G consists of a limited number of members. And a prime p is a factor of its order, then G contains an element whose order is exactly p . Consider G acting through conjugation, the group acts on itself. For $g \in G$, the orbit of g under this action is the conjugacy class of g , denoted as $\text{Orb}(g) = \{xgx^{-1} \mid x \in G\}$.

The stabilizer of g for this action corresponds to the centralizer $C_G(g)$, consisting of all elements in G that commute with g . According to the Orbit-Stabilizer Theorem, the size of g 's orbit under this action is given by $|\text{Orb}(g)| = \frac{|G|}{|C_G(g)|}$, then since p divides $|G|$, consider the action on conjugacy classes where the centralizer's size $|C_G(g)|$ might not be divisible by p . This means the orbit size $|\text{Orb}(g)|$ is divisible by p , implying the existence of a non-trivial conjugacy class whose size is a multiple of p , and thus there is a corresponding element whose order is p .

For the group $G = S_3$ (with 3 order), the order of S_3 is 6. Consider $p = 2$, and S_3 has elements of order 2: (12), (13) and (23). the centralizer of (12) is the subgroup $\{(1), (12)\}$, so $|C_G(12)| = 2$. the orbit of (12) under conjugation is of size $6 \div 2 = 3$. This confirms the existence of a non-trivial conjugacy class, showing that S_3 has elements of order 2. it is same with Cauchy's theorem.

3.3.2. Proof Lagrange's Theorem

Suppose G operates on the set X where $|X| = n$ and the action is transitive. For any $x \in X$, the size of the orbit $G \cdot x$ is $|G : G_x|$, where G_x is the stabilizer of x in G . By the Orbit-Stabilizer Theorem $|G| = |G \cdot x| \times |G_x|$ where $|G \cdot x|$ is the orbit size and $|G_x|$ is the stabilizer size. Both of these sizes divide $|G|$, confirming Lagrange's theorem.

For the group $G = Z_4$ (order 4), G acts on itself by addition. The stabilizer of any element x (say $x = 1$) under this action is trivial, so $G_x = \{0\}$ and $|G_x| = 1$. The orbit of 1 is the entire group $\{0, 1, 2, 3\}$, so the orbit size is 4. Then, by Orbit-Stabilizer Theorem, $|G| = |G \cdot 1| \times |G_1| = 4 \times 1 = 4$. The size of the orbit divides $|G|$, satisfy the Lagrange's theorem.

3.3.3. Proof Burnside's Lemma

When the group G interacts with the set X , the total number of unique orbits is given by the fact that Number of Orbits = $\frac{1}{|G|} \sum_{g \in G} |X^g|$, where $|X^g|$ is the number of elements in X fixed by g . For any $x \in X$, $|X^g|$ is related to the stabilizer G_x , and the theorem that connects the size of the orbit with the stabilizer of an element under group action show the size of each orbit. It can use to count the count of distinct orbits.

For the group $G = Z_2$ acting on the vertices of a square, the count of distinct orbits = $\frac{1}{|G|} \sum_{g \in G} |X^g|$.

The action swaps two vertices at a time. Consider the group action on the vertices $V = \{A, B, C, D\}$ of the square. The stabilizer of any vertex under rotation by 180° is $G_A = \{e\}$, so $|G_A| = 1$. the total number of orbits resulting from the group action is that Number of Orbits = $\frac{1}{|G|} \sum_{g \in G} |X^g|$. So, 2 distinct orbits under this group action.

4. Conclusion

To summarize, this paper discusses the definitions, functions, properties, and claims of the orbit and stabilizer in group actions, as well as the Orbit-Stabilizer Theorem in mathematical group theory. The author also provides the proofs of these theorems. Examples are given to illustrate the order of the stabilizer and the size of the orbit. Finally, the paper explores the relationship between the Orbit-Stabilizer Theorem and three theorems—Cauchy's Theorem, Lagrange's Theorem, and Burnside's Lemma. The paper concludes that orbit and stabilizer are fundamental and foundational concepts in group theory. While the discussion extends from these basic concepts to explore their connections with other theorems, it does not integrate a wide range of examples. In future research can focus on extending the concept of orbits and stabilizers to advanced algebraic structures like Lie groups, algebraic groups, and topological groups. These extensions can reveal new mathematical properties and potential applications in physics, particularly in areas like quantum mechanics.

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