Basic Concept of Isomorphism and its Use in Group Theory and Physics

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Abstract. Isomorphism is a central concept in mathematics and other sciences for defining structures and their equivalencies. The aim of this paper would be to describe the historical commencement of group theory and introduce the basics of concepts in isomorphisms. This paper defines some of the fundamental theorems related to isomorphisms, such as three isomorphism theorems and the epimorphism lemma, whereby the notion of how mathematical objects could have maps onto each other in a structure-preserving way can be defined. It also summarizes the practical applications of isomorphisms in mathematics and physics. Isomorphisms play a huge role in K-theory-a branch of mathematics that deals, in essence, with vector bundles and modules-and in quantum mechanics to further help in understanding symmetries and transformations in physical systems. The importance and implication of isomorphism are very far-reaching in unifying the concepts across scientific fields. This work underscores the importance of isomorphism and related concepts in group theory and other subjects.

Keywords: Isomorphism; Homomorphism; Group Theory.

1. Introduction

Group theory stands as a fundamental mathematical framework in contemporary science. Its roots can be traced back to the late 18th and early 19th centuries, primarily through the work of Joseph-Louis Lagrange and Carl Friedrich Gauss, who explored the symmetries inherent in algebraic equations. In the 1830s, Galois advanced the field by introducing the notion of groups to articulate these symmetries. A significant milestone occurred in 1882 when Dyck's paper provided a pivotal definition of a group via a presentation. By the 20th century, the development and application of group theory had progressed considerably, with notable contributions from mathematicians such as Emmy Noether, who systematically connected group theory with concepts of symmetry and conservation laws [1].

Today, group theory is also regarded as a crucial tool in various disciplines, including physics, chemistry, and computer science. An apparently equally significant isomorphism between quantum mechanics and classical statistical physics has been reported recently demonstrating a link from quantum generating functionals to classical cavity field distribution functions. This connection allows classical techniques to be applied for such quantum-level, nonperturbative computations, which is extremely beneficial from the perspective of computational efficiency. The isomorphism compensates for the lack of classical pictures in quanta, by providing convincing images to treat quantum phenomena like Bose condensation or tunneling. Moreover, grafting the old methods to quantum problems improved computational techniques and made it more palatable for studying some difficult condensed matter physics issues.

This article will introduce both the basic concept of isomorphism and the use of isomorphism in different subjects. Section two will be the definition, three basic theorems and a lemma about isomorphisms. It also contains the definition of homomorphism, which act a crucial role in proving the theorems and the lemma. The three theorems and the lemma are the basic foundation of isomorphism or group action. Section three will be the use of isomorphism in physics and mathematics. Isomorphism can be used in K-theory and quantum mechanics. Section three also points out the significant importance of isomorphism.

2. Definition and Theorem

2.1. Definitions

An isomorphism can be a concept generally used in many fields of Mathematics, including group theory, geometry, etc. This article mainly describes isomorphisms in group theory.

2.1.1. Homomorphism

In most cases, an isomorphism refers to the mapping between two structures. This kind of mapping ensures the structures' properties and operations. To define isomorphism, another important concept homomorphism has to be recognized first:

Let H, G be groups. A function $\phi: H \to G$ is a homomorphism if and only if $\forall h, h' \in H: \phi(hh') = \phi(h)\phi(h')$. This definition forces $\phi(1) = 1$ and has $\phi(h^{-1}) = \phi(h)^{-1}$. For example, let G be a group, and$g \in G$. The map:

$$\mathbb{Z} \to G, n \to g^n. \tag{1}$$

Is a homomorphism.

2.1.2. Isomorphism

With the definition of homomorphism, it is easier to introduce isomorphism: A homomorphism $\phi: H \to G$ is an isomorphism if and only if it is bijective.

The definition laid out a lemma. Let $\phi: H \to G$ be an isomorphism. Then the inverse function $\phi^{-1}: H \to G$ is also an isomorphism.

Proof. Take $h, h' \in H$. As ϕ is surjective, there exist $g, g' \in G$ having $\phi(g) = h$ and $\phi(g') = h'$. For ϕ is a homomorphism, $\phi(gg') = hh'$ and so $\phi(hh') = gg' = \phi^{-1}(h)\phi^{-1}(h')$.

2.1.3. Isomorphic Groups

Isomorphic Groups is an important concept bringing out by isomorphism. It occurred great importance in group theory which is used to describe the equivalence of two groups in term of their structure.

When Groups G and H are isomorphic, which can be written as $G \simeq H$, if and only there exists an isomorphism $\phi: G \to H$. Isomorphic groups can be thought as being the same as groups.

2.2. Theorems

2.2.1. First Isomorphism Theorem

Statement. Let $\phi: G \to H$ be a group homomorphism, then the quotient group $G/\ker\phi$ is isomorphic to the image group $Im(\phi)$. or in other words, $G/\ker\phi \simeq Im(\phi)$.

Proof: First define a map by $G/\ker\phi \to Im(\phi)$ by $\psi(g\ ker\phi) = \phi(g)$.If $g_1 ker\phi = g_2 ker\phi$, Then:$g_1^{-1}g_2 \in ker\phi$ so $\phi(g_1^{-1}g_2) = e_H$.therefore,$\phi(g_1) = \phi(g_2)$ hence have $\psi(g_1\ ker\phi) = \psi(g_2\ ker\phi)$.

For $g_1\ ker\phi, g_2\ ker\phi \in G/ker\phi$, $\psi((g_1\ ker\phi)(g_2\ ker\phi)) = \psi(g_1 g_2 ker\phi) = \phi(g_1 g_2) = \phi(g_1)\phi(g_2) = \psi(g_1\ ker\phi) \cdot \psi(g_2\ ker\phi)$ so ψ is a homomorphism

Then proving injectivity and subjectivity. Suppose $\psi(g\ ker\phi) = e_H$. This means $\phi(g) = e_H$, so $g \in ker\phi$, and hence $g\ ker\phi = ker\phi$. thus ψ is a homomorphism. For any$h \in Im(\phi)$, there exists some $g \in G$ such that $\phi(g) = h$. Thus, $\psi(g \ker\phi) = \ker\phi$. thus, $\psi(g\ ker\phi) = h$, so ψ is surjective. Therefore, ψ is a bijective homomorphism, so? $G/\ker\phi \simeq Im(\phi)$.

2.2.2. Second Isomorphism Theorem

Statement. Let G be a group and have H be a subgroup of G and let N be a normal subgroup of G. There exists an isomorphism:$H/(H \cap N) \simeq HN/N$ $HN$is the subgroup generated by H and N.

Proof. Since N is normal in G and H is a subgroup of G, $(H \cap N)$ is normal in H as for any $h \in H, h(H \cap N)h^{-1} = H \cap N$. Then define a map: $\psi: H \to (HN)/N$ by $\psi(h) = hN$. For $h_1, h_2 \in H, \phi(h_1 h_2) = (h_1 h_2)N = h_1 N \cdot h_2 N = \psi(h_1) \cdot \psi(h_2)$. Thus, ψ is a homomorphism.

Then proving bijiectivity, If $\psi(h) = N$, then $h \in N$ and implies $h \in H \cap N$. Hence, $\ker\psi = H \cap N$. The injectivity has been proved. For any $xN \in (HN)/N$, $x \in HN$. $x = hn$ Where $h \in H$ and $n \in N$, thus $\psi(h) = hN = xN$ and ψ is surjective. Hence, ψ is bijiective? Then it is proved that ψ is a bijective homomorphism, so $H/(H \cap N) \simeq HN/N$.

2.2.3. Third Isomorphism Theorem

Statement. Let G be a group. Let M be a normal subgroup of G, and let N be a subgroup of M that is also a normal subgroup of G. Then the quotient group G/N is a group, and there exists an isomorphism:

$$(G/M)(G/N) \simeq G/N \tag{2}$$

Proof. Consider the natural projection map $\pi: G \to G/N$ given by $\pi(g) = gN$, and the projection

Map $\phi: G \to G/M$ given by $\phi(g) = gM$. Then define a map ψ from G/M to G/M as follows:

$\psi(gM) = gN, g \in G$. Then the author will prove homomorphism. For any $gM, hM \in \frac{G}{M}$, one has $\psi((gM) \cdot (hM)) = \psi\big((gh)M\big) = (gh)N$. Thus, it is inferred that $\psi(gM) \cdot \psi(hM) = gN \cdot hN = (gh)N$. Therefore, $\psi((gM) \cdot (hM)) = \psi(gM) \cdot \psi(hM)$ which shows that ψ is a homomorphism. Then prove bijectivity. First, to show that ψ is injective, consider the kernel of ψ. If $\psi(gM) = N$, then $gN = N$, meaning $g \in N$. Therefore, $gM \in N/M$, so $\ker\psi = N/M$. Hence the kernel is a normal subgroup of G/M. By the First Isomorphism Theorem for groups, ψ includes an isomorphism: $(G/M)(G/N) \simeq Im(\psi)$. The image of ψ is G/N, hence: $(G/M)(G/N) \simeq G/N$. Secondly prove subjectivity. For any $gN \in G/N, \psi(gM) = gN$, showing that every element in

G/N Is the image of some elements in G/M under ψThus, the bijectivity of ψ is proved. Hence, ψ is an isomorphism?

2.2.4. Lemma

Statement. Let $f: G \to H$ be an be an epimorphism between groups G and H. Let N be a normal subgroup of G. Then the image of N under f is a normal subgroup of H, and the following isomorphism holds: $(G/N) \simeq (Im(f)/Im(f \circ Inj_N))$. Where Inj_N is the inclusion map of N into G, and $Im(f \circ Inj_N)$ is the image of N under f, which is a normal subgroup of $Im(f)$.

Proof. Given $f: G \to H$ is a surjective homomorphism and N is a normal subgroup of G, define the map m from G/N to $Im(f)/Im(f) \cap f(N)$ by: $m(gN) = f(g) \cdot \big(Im(f) \cap f(N)\big)$, where gN represents the coset of N in G. Then prove m is a homomorphism.

Let $m\big((g_1N) \cdot (g_2N)\big) = m(g_1N) \cdot m(g_2N)$. In$G/N$, the product $(g_1N) \cdot (g_2N)$ is $(g_1g_2)N$. Applying m, one gets $m((g_1g_2)N) = f(g_1g_2) \cdot \big(Im(f) \cap f(N)\big)$. On the other hand, $m(g_i) = f(g_i) \cdot \big(Im(f) \cap f(N)\big), i = 1,2$. Thus, $m(g_1N) \cdot m(g_2N) = f(g_1) \cdot \big(Im(f) \cap f(N)\big) \cdot f(g_2) \cdot \big(Im(f) \cap f(N)\big) = f(g_1)f(g_2) \cdot \big(Im(f) \cap f(N)\big) = f(g_1g_2) \cdot \big(Im(f) \cap f(N)\big)$. Thus, m is a homomorphism.

Then the author will prove m is bijective. Firstly, prove injectivity. Suppose $m(gN) = m(hN)$. Then $f(g) \cdot Im(f) \cap f(N) = f(h) \cdot Im(f) \cap f(N)$. This implies that $f(g)^{-1}f(h) \in Im(f) \cap f(N)$. Hence, $f(g)^{-1}f(h) = f(n)$ for some $n \in N$ $f(g^{-1}h) = f(n)$ for some $n \in N$ Since $f(g^{-1}h) \in f(N)$, it follows that $g^{-1}h \in N$, so $gN = hN$. Therefore, m is injective. Then, for any $x \cdot \big(Im(f) \cap f(N)\big) \in Im(f)/\big(Im(f) \cap f(N)\big)$, where $x \in G$, have $m(xN) = f(x) \cdot \big(Im(f) \cap f(N)\big)$. Therefore, m is surjective. Hence, it's bijective and it's an isomorphism.

3. The Use of Isomorphism

3.1. Use in Group Action

Isomorphism is crucially important in classifying finite groups. Groups can be categorized into known structures by identifying the isomorphisms. At the same time, isomorphisms can be related to the structure of subgroups in order to simplify complex problems.

Isomorphism studies how algebraic structures (such as groups and rings) can be represented using linear transformations and matrices. In terms of group actions, a group acts on a set, when the elements are permuted and this action can be shown through linear representations. Isomorphism says that there is a correspondence between two encodings which represents both to be effectively equivalent in behavior and properties. Two isomorphic representations of a group are structurally the same, thus allowing mathematicians to classify them and thereby reducing their study. Such an equivalence is especially important in application to physics, as understanding symmetries of physical systems often requires analyzing the group actions and their representations [2].

Isomorphisms facilitate the comprehension of equivalent actions. For example, the configuration of orbits and stabilizers in group actions can be clarified through the use of isomorphism, aiding in the examination of the group's influence on various sets. In the context of symmetry analysis, isomorphism plays a crucial role in recognizing the equivalence among distinct symmetric structures, which is essential for addressing symmetry-related problems.

3.2. Use in Other Mathematical Theory

According to F. T. Farrell and L. E. Jones [3], they propose conjectures related to the isomorphisms between different algebraic structures. Specifically, they conjecture that certain n-spectra associated with a topological space X can be computed from the n-spectra associated with the covering spaces of X, which have simpler fundamental groups. This is based on the idea that there should be a simple way to relate the algebraic K-theory of a space to the algebraic K-theory of its covering spaces. Isomorphisms between different algebraic structures associated with group rings are essential for understanding the algebraic K-theory of these groups.

Isomorphism is one of the basic and challenging concepts for Group Isomorphism (GpI) problems taking into consideration non-abelian p-groups with odd primes, since p-group-isomorphic or not does matter a-lot in computational representation. The application of isomorphism in this study encompasses [4].

For the search to decision and counting. Decision reduction for GpI degrees which are essential for example to analyze complexity of isomorphism problems. Hence, if one can decide whether two groups are isomorphic (the decision problem), then the related problems arise in a natural fashion: given such an algorithm to recognize group theoretic properties, one could also have algorithms that search for and count automorphisms of two finitely presented groups.

3.3. Use in Physics

Physicists use isomorphisms to understand that two different looking systems are actually related through the same symmetry structure. For example, the collection of three dimensional rotations SO (3) is related to the group SU(2), which is one of the most influential in quantum mechanics is SU(2). The fact that SU (2) is a double cover of SO (3) these spaces can (are homomorphic) to SO (3) - meaning spin — the intrinsic angular momentum of elementary particles, is a quantum propery associated with classical rotational symmetry [5].

Two images that differ by a reflection result in the model Hamiltonian also being equal to its own complex conjugate (the system exhibits PT symmetry). For example, if that particular symmetries of a quantum system are represented by a group isomorphic to some subgroup of the Lorentz Group, it may be useful when attacking related QFT problems.

Isomorphisms between Conservation Laws: Conservation laws are sometimes explained by isomorphisms of groups. The conservation of electric charge — for instance, can be the phase after the volume integration has been performed. The same goes for U (1) symmetry group, and the connection this has to other groups sheds light on how different forces like electromagnetism get unified in physics via their gauge symmetries (Standard Model).

The symmetries again, are isomorphisms between its groups and through these physical theories relate to each other. Even special relativity and electromagnetism are perfectly aligned in the Poincaré group. One reason that isomorphisms between alternative representations are important is being able to see the deep connections underlying these theories.

The isomorphism offers a new perspective on quantum phenomena such as Bose condensation and quantum tunneling. By treating quantum exchange in terms of classical chemical equilibria and using concepts like the classical law of mass action, the paper provides insights into the nature of quantum exchange and Bose condensation. This approach helps to visualize quantum effects, such as tunneling, in terms of classical analogs like solitonic configurations or flexible chain molecules [6].

The connection between quantum and classical theories facilitated by the isomorphism allows for the use of classical methods, including those related to the renormalization group (RG) technique, to solve quantum mechanical problems. This offers a powerful framework for obtaining quantitative solutions to condensed matter problems that were previously difficult to address with conventional quantum methods.

3.4. Significant Importance of Isomorphism

As experimental studies conducted in laboratory settings have found that individuals often lack awareness of relationships when faced with isomorphic problems [6]. However, it is important to consider certain limitations associated with this study when interpreting the results. Participants frequently demonstrate a lack of motivation and typically engage with abstract mathematical tasks for only brief periods. This limited engagement restricts their ability to develop a deeper structural understanding of the mathematical concepts being introduced. The short duration of interaction may impede the participants' capacity to fully comprehend and internalize the foundational principles, ultimately impacting the overall effectiveness of the learning experience.

This scenario highlights a notable deficiency in the existing mathematics education framework, where the fleeting engagement with abstract tasks fails to sufficiently facilitate the development of a thorough and meaningful understanding of mathematical structures. To address this concern, it may be necessary to implement strategies aimed at boosting participant motivation and prolonging their engagement with mathematical tasks, thereby promoting a more profound and lasting comprehension of the subject matter.

4. Conclusion

Isomorphism may be a basic building block in mathematics and physics, tending to a powerful tool for the understanding of complicated structures. The paper has discussed isomorphism, mainly concerning group theory, and traced it from the early contributions by Lagrange, Gauss, and Galois up to the giant leap caused by Noether's work, connecting group theory with symmetry and conservation laws. It should be pointed out that the discussion of three major theorems and one important lemma underlines the mathematical rigor and significance of isomorphism in these areas. The wide applicability of isomorphism, especially in the bridging of geometric divides across wide fields such as mathematics and physics, is of great and critical value to scientific study. Isomorphisms, according to the likeness in structure for systems seemingly different, allow much better problem-solving and theoretical development. While this paper has shown, isomorphism is not strictly a theoretical conception, but rather a key term that has continued and will continue to impress its mark on scientific understanding.

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