Applications and Related Concepts of Orbit-Stabilizer Theorem

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Abstract. As the essential part of abstract algebra, group theory takes a critical place on the field of pure mathematics. Approaching the study of it from the aspect of group action is a great starting point. This paper will firstly expound several related concepts and theorems, such as stabilizer, orbit, fixator, conjugacy class, center of group and centralizer, and the Lagrange's Theorem. Based on that, the paper investigates three essential theorems themselves and their proofs, which are Orbit-Stabilizer Theorem, Class Equation, and Burnside's Lemma. This paper specifically provides wonderful insights about the applications of the Orbit-Stabilizer Theorem. The theorem is directly utilized in the derivations of Burnside's Lemma and the Class Equation, also in the deduction of the general formula of the order of symmetric group S_n. There are always some connections between the concepts in these theorems and examples with the orbit and stabilizer, thus Orbit-Stabilizer Theorem can perform brilliantly. This paper provides wonderful insights that effectively connect the Orbit-Stabilizer Theorem with other parts of Group Theory, also brings some new ideas to solving problems and prove theorems.

Keywords: Orbit-Stabilizer Theorem, Group Action, Class Equation, Burnside's Lemma.

1. Introduction

In abstract algebra, group theory takes an essential and important place. It studies the binary operation, sets, elements and the connections between them. The group concepts are derived gradually from several generations of mathematicians, and the conclusions of permutation group were firstly acquired from the research of the general solutions of high degreed polynomial equations by Lagrange, Ruffini and so forth [1]. Évariste Galois historically created the term "group" and built up a theory about the connection between the emerging group theory and field theory, which is known as Galois Theory now.

Other algebraic structures such as field, ring could be constructed and developed further based on the evolution of group theory, which provides a series of tools and methods that can be used in other areas of mathematics. For example, algebraic topology uses those algebraic objects to investigate topology and the algebraic number theory, which relate with abstract algebra and even act as a crucial part of the proof of Fermat's Last Theorem. In addition, on account of the group theory, there are dramatic influences on the progress of physics, chemistry and biology, also the computer science and cryptography for the modern society [2].

This paper would focus on the study of group action of finite group with the applications of Orbit-Stabilizer Theorem, which are both vital part of finite group theory. The content would not only includes some related definitions, lemmas and theorems, but also numerous wonderful applications of the knowledge that discussed in the paper. Firstly, the concepts of orbit, stabilizer and the group action would be introduced, and the paper will bring in a famous frequently-used theorem, Lagrange's Theorem, which will be comprehended as a preparation of the proof Orbit-Stabilizer Theorem in section 3. Moreover, in section 3, an example that derives the general formula of the order of symmetric group S_n would complete by a method that uses Orbit-Stabilizer Theorem. Also, some concepts about Class Equation, such as, conjugacy class, centralizer and center of a group, which would be elaborated before a more effective and elegant proof for Class Equation compared with predecessors' proof, and this proof is derived by the Orbit-Stabilizer Theorem. Finally, the proof of Burnside's Lemma would directly finish by using the Orbit-Stabilizer Theorem, and also some general fact based on the study of orbits.

2. Background Knowledge and Method

2.1. Basic Concepts

Group Action is quite vital in mathematical world. It is a crucial foundation of several famous theorems and other Group Theory concepts. Also, there are some concepts that relate with group action with interesting and profound properties.

Definition 2.1 [3]: If a finite group G acts on a finite set S, then ordered pair (g, s) is mapped into $g * s$ from $G \times S$ to S. Such mapping $\phi: G \times S \rightarrow S$ is referred to as the group action of group G on S $(\phi(g, s) = (g * s))$, and this action satisfies following 2 properties. Property 1: $\forall s\ \phi(e, s) = e * s = s$ and e act as the identity. Property 2: $\phi(gg', s) = \phi(g, \phi(g', s))$, where $g, g' \in G$.

Corollary 2.2 [4]: Since the group action sends the elements in S to S, so the mapping ϕ is evidently a permutation.

Moreover, based on the group action, several interior concepts have been developed, and those concepts would be used frequently on the proofs and expressions of latter theorems and lemmas.

Definition 2.3 [4]: Under the group action ϕ, the elements in G such that

$$\{\, g \in G : \phi(g, s) = s\} \tag{1}$$

Which is known as the stabilizer of s in G (all the $g \in G$ that fixates s), denoted by $Stab_G(s)$

Definition 2.4 [4]: Elements inside the S such that

$$\{\, \phi(g, s) : g \in G\,\} \tag{2}$$

Is called orbit of s (all the images of s), denoted by $Orb_G(s)$.

Definition 2.5 [5]: Elements inside S such that:

$$\{\, \phi(g, s) = s : s \in S\,\} \tag{3}$$

Is called the fixator of s in G (all the $s \in S$ that fixated by g), denoted by $fix_G(g)$

Definition 2.6 [4]: For a subgroup H of G and the elements inside G, the subset $\{g * h: h$ inside $H\}$ is H's left coset in G, denoted by gH . The right coset Hg denote the set $\{h * \mathrm{g}: \mathrm{h}$ inside $H\}$. Also, the set of every left cosets of H in G is written G/H.

Definition 2.7 [5]: The symbol $[G: H]$ represents the index of the subgroup H in G, which **represents how many left cosets of H are there, it also equals to the number of the right cosets.**

2.2. Basic Conclusions

Lemma 2.8 [4]: The stabilizer of any elements in S constitutes a subgroup in group G.

Related with the index of a subgroup, there would be a quite essential theorem called Lagrange's Theorem, which appears in many proofs of other theorems.

Theorem 2.9 [5]: *(Lagrange's Theorem)* for a finite group G, there exists a subgroup H. Then $|G|$ could be a multiple of $|H|$ and the product of index of H and the order of H is equals to the order of G.

It is quite unsophisticatedly to prove this theorem, since the left cosets partition the group G and there are bijections between the elements in H with the elements in left cosets. So, it is obviously that one has the product of the order of set of left cosets. The order of H is equals to the order of G.

3. Results and Applications

3.1. Orbit-Stabilizer Theorem

This profound mathematical result illustrates how the order of a finite group G could be related with the stabilizers and orbits. This paves a bright way for other applications and conclusions about group actions and group structures.

Theorem 3.1 [6]: *(Orbit-Stabilizer Theorem)* Consider a finite group G acting on a collection of elements inside S, for an arbitrary element in set S, the number of elements in G can be represented as follow:

$$|G| = |Orb_G(s)| \cdot |Stab_G(s)| \tag{4}$$

Proof. [4] Note the Lemma 2.8 states that $Stab_G(s)$ is a subgroup of G, and one can find the relation between the orders of G and it is subgroup by Theorem 2.8. That is, it is found that $|G| = [G{:}\,Stab_G(s)] \cdot |Stab_G(s)|$. The only next step is to prove that $[G{:}\,Stab_G(s)] = |Orb_G(s)|$.

Let two arbitrary elements $a, b \in Stab_G(s)$. Since $a * s = b * s = s$ that conclude from the Definition 2.3. Then there is a mapping of the left cost of $Stab_G(s)$ to $Orb_G(s)$, i.e.

$$(g * a) * s = g * \big(a * (s)\big) = g * s = g * \big(b * (s)\big) = (g * b) * s \tag{5}$$

For the place where s is sent, which is completely depending on which g in G is picked, and a certain g can contribute a left coset of $Stab_G(s)$ by combined with a $Stab_G(s)$. Since g sends s to a unique $g * s$, then there obviously exists left coset $g * Stab_G(s)$ that is matching with an element $g * \mathrm{s}$ in $Orb_G(s)$, hence this mapping is surjective. Next, consider two elements g and h inside G, such that they send s to the same place. Hence, $h^{-1} * g * s = s$ which lead to the element $h^{-1} * g$ belongs to the stabilizer of s, so that the left cosets of stabilizer of s under the elements g and h are equal. Thus, the mapping is bijective, and the Eq. (5) is derived as the proof of this theorem is done.

Example 3.2. Consider the permutation σ in a symmetric group S_n that send elements from the set of number between 1 to natural integer n to the set itself, where 1 is sent to s_1, 2 is sent to s_2 and so forth, so that s_k is sent to k for $k \in \{1,2, \dots n\}$, such that

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ s_1 & s_2 & s_3 & \cdots & s_n \end{pmatrix} \tag{6}$$

Then without loss of generality, take 1 to see its orbit and stabilizer, and their orders, try to use Theorem 3.1 to find $|S_n|$. $Orb_{S_n}(1) = 1,2, \dots (n-1), n$, where it is obvious that $\left|Orb_{S_n}(1)\right| = n$. $Stab_{S_n}(1)$={all the permutation in G that fixes 1}. To find the order of this stabilizer is equivalent to find how many permutations in G are fixes 1. Let all this kind of permutations form a subgroup α_2, so $Orb_{\alpha_2}(2) = \{\, 2, \dots (n-1), n\}$, and the order or it is simply $n-1$. Then, repeat the procedure to let the permutations in G that fixes the elements 1 to k $(k \in Z^+)$ be the subgroup α_k of S_n, until $k = n$. Thus, for the orbit of k in such subgroup α_k, has the order of $(n-k+1)$. After that, $\left|Orb_{\alpha_n}(n)\right|$ and $\left|Stab_{\alpha_n}(n)\right|$ are simply 1, by applying Orbit-Stabilizer Theorem, $|\alpha_n| = 1$, so:

$$|\alpha_{n-1}| = \left|Orb_{\alpha_{n_1}}(n-1)\right| \cdot \left|Stab_{\alpha_n}(n)\right| = \left|Orb_{\alpha_{n_1}}(n-1)\right| \cdot |\alpha_n| = (n-(n-1)+1) \cdot 1 = 2 \tag{7}$$

Similarly $\left|\alpha_{n_2}\right| = (n-(n-2)+1) \cdot 2 = 3!$ $\quad \left|\alpha_{n_3}\right| = 4!$, and so forth. It's obvious that $\left|\alpha_{n_k}\right| = (k+1)!$, so that $\left|\alpha_{n_2}\right| = \left|Stab_{S_n}(1)\right| = (n-1)!$. Thus $|S_n| = |Orb_{S_n}(1)| \cdot |Stab_{S_n}(1)| = n \cdot ((n-1)!) = n!$, and it is same for using the orbits and stabilizers of other elements in the set 1 to n initially. This provides another useful method to find the order of S_n, rather than just deriving the general formula by some combinatory tools.

3.2. Class Equation

The Class Equation in group theory provides another method to find the number of elements that in a group G. It relates with some concepts about the center of G, centralizer, and the conjugacy class. Also, its proof can be a done by the application of Orbit-Stabilizer Theorem.

Definition 3.3 [7] All the elements inside the group G that satisfy the commutative law with all the other elements in G, such that

$$Z(G) = \{\, g \in G : \forall g' \in G, g \cdot g' = g' \cdot g\} \tag{8}$$

Is called the center of G, denoted as $Z(G)$.

Definition 3.4 [7] All the elements inside the group G that satisfy the commutative law with element g' inside G, such that

$$C_G(g') = \{g \in G : g \cdot g' = g' \cdot g, g' \in G\}. \tag{9}$$

This is known as the centralizer of g' in G.

Definition 3.5 [7] Consider three elements g, g', g'' that are the elements inside G, then let g' and g'' be conjugate. For g', its conjugacy class is denoted by $Cl_G(g')$ such that:

$$Cl_G(g') = \{g \cdot g' \cdot g^{-1} : g \in G\} \tag{10}$$

As a matter of fact, conjugacy class is an equivalence class, where this class of an element s in set S contains all the elements in S that have equivalence relation to s. In addition, the equivalence relation follows the three properties: Reflexivity, Symmetry, and the Transitivity.

Consider the example of the conjugacy relation in group G as the equivalent relation that $\phi(g, g')$ represents g and g' are conjugate. Then g and g' are called equivalent on conjugacy relation. Thus, if some elements are conjugate with each other, then they would all in one same conjugacy class, or in another words, the conjugacy classes of those elements are identical. Similarly, for the entire group action, its orbits are actually the equivalence classes, so all these orbits would partition the set that the group act on as well.

Lemma 3.6 [8] for a group G, its conjugacy classes are partitions of G, where the union of those classes constitute the group G:

$$G = \bigcup_{g \in G} Cl_G(g) \tag{11}$$

Lemma 3.7 [7] For conjugacy classes $Cl_G(g_i)$ in the group G, $|Cl_G(g_i)| = [G : C_G(g_i)]$, where i is between 1 and n inclusively and the n represents the number of conjugate classes in G.

Proof [7] Proving the order of $Cl_G(g_i)$ equals to the index of centralizer of g_i in G. It can be approached by proving there is a bijection between these two sets.

Define a function φ that map elements in $Cl_G(g_i)$ to the set $G/C_G(g_i)$, such that $\varphi(g \cdot g_i \cdot g^{-1}) = g \cdot C_G(g_i)$. Firstly, to prove φ is a well-defined function, one can consider two elements $j, k \in G$ such that $j \cdot g_i \cdot j^{-1} = k \cdot g_i \cdot k^{-1}$. Therefore,

$$(j^{-1} \cdot k) \cdot g_i \cdot (k^{-1} \cdot j) = g_i \Rightarrow j^{-1} \cdot k \in C_G(g_i) \Rightarrow k \cdot C_G(g_i) = j \cdot C_G(g_i) \tag{12}$$

Now try to prove the function φ is bijective. For $j \cdot C_G(g_i) = j \cdot C_G(g_i)$, it is found that

$$j^{-1} \cdot k \cdot C_G(g_i) = C_G(g_i) \Rightarrow j^{-1} \cdot k \in C_G(g_i) \Rightarrow j \cdot g_i \cdot j^{-1} = k \cdot g_i \cdot k^{-1} \tag{13}$$

Thus, φ is an injective function, also, for each element $\varphi(g \cdot g_i \cdot g^{-1})$ in $G/C_G(g_i)$, it is sent by the corresponding element $g \cdot g_i \cdot g^{-1}$ in $Cl_G(g_i)$, hence φ is surjective. Consequently, φ is bijective, then $|Cl_G(g_i)| = |G/C_G(g_i)| = [G : C_G(g_i)]$.

Lemma 3.8. For every finite group G, there always exist a group action on G itself by conjugation.

Proof. It is obvious that it would be a bijection exists from group G to itself, and to check the Property 2 of group action, one finds that

$$\phi(g, \phi(g', g'')) = g(g'g''g'^{-1})g^{-1} = (gg')g''(g'^{-1}g^{-1}) = (gg')g''(gg')^{-1} = \phi(gg', g'') \tag{14}$$

Thus, for every finite group G, one can find a group action on itself by conjugation.

As the matter of fact, the proof for Lemma 3.7. Above could be sort of complicated to think of and utilize, since Lemma 3.8 showed the existence of group action on itself by conjugation for all finite group G. It is possible to use the Orbit-Stabilizer Theorem to realize a more effective version of the proof to complete the proof of Class Equation later by the application of Orbit-Stabilizer Theorem**.**

Proof of Lemma 3.8. Consider the action that on group itself by conjugation, where the group action is

$$\phi(g,g') = g * g' = g \cdot g' \cdot g^{-1} \tag{15}$$

Notice that under the group action, for an arbitrary element g' inside the G, it is actually true that $Orb_G(g') = Cl_G(g')$, also $Stab_G(g') = C_G(g')$.

By applying Orbit-Stabilizer Theorem (Theorem 3.1) and Lagrange's Theorem (Theorem 2.9.), it is inferred that

$$|Orb_G(g')| = \frac{|G|}{|Stab_G(g')|} \Leftrightarrow |Cl_G(g')| = \frac{|G|}{|C_G(g')|} = [G : C_G(g')] \tag{16}$$

Thus, by Lemma 3.9, the Eq. (16) is true for all finite group G.

Theorem 3.10. [9] *(Class Equation)* for the representatives of non-trivial distinct conjugacy classes $g_1, \dots g_n$, where g_i corresponds to $Cl_G(g_i)$ for $1 \le i \le n$, it is found that

$$|G| = |Z(G)| + \sum_{i=1}^{n} [G : C_G(g_i)] \tag{17}$$

Proof. Since the conjugacy classes partition the group by Lemma 3.6, so for a group G with its non-trivial conjugacy classes $Cl_G(g_1), \dots, Cl_G(g_n)$, and the trivial conjugacy classes $z_1, \dots z_m$ which are the centers of group, where n and m are positive integer, one can infer that

$$|G| = |z_1| + \dots + |z_m| + |Cl_G(g_1)| + \dots + |Cl_G(g_n)| \tag{18}$$

Next, since each trivial conjugacy classes only have one element in it, and all those elements are centers of the group, so the sum of first m term are simply the $|Z(G)|$. Also, one can write those expression in the summation form

$$|G| = |Z(G)| + \sum_{i=1}^{n} |Cl_G(g_i)| \tag{19}$$

Now, the Eq. (19) is quite similar with the Class Equation. By using the Lemma 3.8 that discussed previously, $|Cl_G(g_i)| = [G : C_G(g_i)]$ for $1 \le i \le n$, thus:

$$|G| = |Z(G)| + \sum_{i=1}^{n} [G : C_G(g_i)] \tag{20}$$

The Class Equation is derived. This proof is a wonderful application of Orbit-Stabilizer Theorem, which allows for more group theory problems and having deeper connection with the group action and promoted by related theorem.

3.3. Burnside's Lemma

The Burnside's Lemma is a fundamental theorem that connects the bridge of group theory and combinatory. It is also an application of Orbit-Stabilizer Theorem and other relevant concepts. There are some theorems and lemma derived from it as well, such as Pólya Enumeration Theorem.

Theorem 3.11. [10] *(Burnside's Lemma)* Consider a finite set S and a group G with ϕ as action of G on S, it is found that

$$|Orb_G(\phi)| = \frac{1}{|G|} \sum_{g \in G} |fix_G(g)| \tag{21}$$

Proof. Try to find a way to utilize the Orbit-Stabilizer Theorem to promote the proof. Note that:

$$\sum_{g \in G} |fix_G(g)| = \sum_{s \in S} |Stab_G(s)| = \sum_{s \in S} \frac{|G|}{|Orb_G(s)|} \tag{22}$$

Then by the direct usage of the Orbit-Stabilizer Theorem. Those summation of reciprocals of the orders of orbit should be simplified further, since the orbits in $Orb_G(\phi)$ partitioned the set S, so it is possible to destruct the summation of reciprocals of the orders of orbit to find the summations of the reciprocals of the orders of orbit of elements in one orbit, and then sum those summations:

$$|G|\cdot\sum_{s\in S}\frac{1}{|Orb_G(s)|}=|G|\cdot\sum_{Orb_G(s)\in Orb_G(\phi)}\sum_{s\in Orb_G(s)}\frac{1}{|Orb_G(s)|} \tag{23}$$

Recall that the orbit class is an equivalence class, so for all $s\in Orb_G(s)$, $|Orb_G(s)|$ are the same number, and there are $|Orb_G(s)|$ numbers of reciprocals of the order of orbits in the summation, thus:

$$|G|\cdot\sum_{Orb_G(s)\in Orb_G(\phi)}\sum_{s\in Orb_G(s)}\frac{1}{|Orb_G(s)|}=|G|\cdot\sum_{Orb_G(s)\in Orb_G(\phi)}\frac{|Orb_G(s)|}{|Orb_G(s)|} \tag{24}$$

Then put the summation of the order of fixator and the simplified summation in a same equation:

$$\sum_{g\in G}|fix_G(g)|=|G|\sum_{Orb_G(s)\in Orb_G(\phi)}1=|G|\cdot|Orb_G(\phi)| \tag{25}$$

Finally, by rearrange the Eq. (25), the lemma is proved by using Orbit-Stabilizer Theorem that was proved previously.

4. Conclusion

This paper clarifies the concept of group action and other related concepts in finite group theory, including the basic definitions of orbit, stabilizer, and group action. The essential usage of Lagrange's Theorem and the lemma about subgroup stabilizer can provide a great idea of the proof to the Orbit-Stabilizer Theorem, while readers are able to appreciate the ingenious thought of group theory. Furthermore, the Class Equation is basically concluded from the understanding of the centralizer, conjugacy class and the center of group, after this deduction, the Orbit-Stabilizer Theorem plays an important part to complete the proof, which showed the summation of orders of the conjugacy classes equals the summation of the orders of the set of left cosets of the centralizers of representatives of conjugacy classes in the group, which provided a convenient and effective way to further prove the Class Equation. The Orbit-Stabilizer Theorem is also used directly to prove the Burnside's Lemma, which is started from the facts that the sum of the orders of fixators is equals to the sum of the orders of stabilizers, and the orbits partition the set, thus the equation about order of stabilizer can became helpful to finish the proof of Burnside's Lemma. Besides the ideas and methods this paper provided, there are still unknown aspects that can be considered to lead to more extensive applications are waiting to be deliberated on. This paper would be one of the parts that promote the further development of the research of group action and theorems in group theory in the future.

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