Laplace Theorem for hyperbolic complex determinants

Haitao Ye *, Jie Zhang, Shiyu Fu

Department of Mathematics, Shaoxing University, Shaoxing, China, 312000

* Corresponding Author Email: yht2740599542@foxmail.com

Abstract. This paper studies the Laplace theorem for hyperbolic complex determinants, aiming to explore the algebraic properties of hyperbolic complex numbers and their applications in analysis. The research significance lies in laying the foundation for the algebraic theory of hyperbolic complex analysis and promoting research development in physics and related fields. The research methods include defining hyperbolic complex numbers and their operational rules, decomposing hyperbolic complex determinants into two real determinants, and further proving the Laplace theorem for hyperbolic complex matrices. The research results indicate that hyperbolic complex determinants possess unique decomposition properties, and the Laplace theorem is applicable to hyperbolic complex matrices, providing an effective method for calculating high-order determinants and partitioned matrix determinants. The discussion section emphasizes the theoretical innovations of this research, such as proposing new definitions of hyperbolic complex algebraic structures, as well as methodological innovations, such as providing a concise method for calculating high-order determinants. Furthermore, it anticipates the potential application value of the research results in fields such as physics and hyperbolic complex analysis.

Keywords: Hyperbolic Complex Numbers, Determinant, Laplace Theorem.

1. Introduction

This paper investigates affine transformations on the hyperbolic number plane, deriving formulas and properties for translations, rotations, reflections, and special reflections using hyperbolic numbers. The contributions include novel results on affine transformations in the Lorentzian plane and applications to generating hyperbolical fractals through iterated function systems [1]. This paper investigates the Riemann-Hilbert boundary value problem for generalized regular functions in multiple hyperbolic complex numbers on a cylindrical domain. The contributions include obtaining the existence and integral representation of the solutions to this boundary value problem [2]. This paper derives hyperbolic complex algebraic structures in dimensions two, three, and four, extending systematic approaches previously developed for other complex algebraic structures. The contributions include defining hyperbolic vector products, powers, and exponential functions, as well as discussing hyperbolically holomorphic functions and modified Cauchy-Riemann differential equations [3]. This paper delves into the geometric, algebraic, and analytical aspects of bicomplex Möbius transformations, offering novel insights and proofs related to these transformations. By analyzing bicomplex numbers as the sum of hyperbolic numbers, it presents a comprehensive understanding of bicomplex Möbius transformations and their geometric implications, including the introduction of new concepts such as the bicomplex cross ratio and Lobachevsky's bicomplex geometry [4].

This paper presents several theorems related to Laplace transforms, including power theorems and autocorrelation theorems for both one-dimensional and two-dimensional cases. It also discusses the residue theorem and differential properties of two-dimensional Laplace transforms, contributing to the understanding and application of Laplace transforms in solving differential equations and computing integrals [5]. This paper generalizes and proves Laplace's theorem, discussing its application to arbitrary k-order submatrices, and further demonstrates its use in proving determinant products. It also applies Laplace's theorem to the calculation of higher-order determinants and partitioned matrix determinants, providing a more general formula for partitioned matrix determinant calculations [6]. This paper proposes a novel approach to introduce the concept of determinants and presents concise proofs for determinant properties and Cramer's rule, making the theory of determinants easier to teach and learn. The key contributions are the streamlined introduction of determinant concepts and the simplified proofs, which enhance student comprehension and foster independent thinking and research capabilities, providing a valuable example for mathematics textbook reform [7]. This paper presents novel methods for finding matrix inverses and determinants by studying transition matrices between bases in linear spaces. The key contributions are the discovery of inverses for specific matrix types and the resolution of the inverse and determinant of Vandermonde matrices, contributing to advancements in matrix theory and linear algebra [8].

This paper demonstrates the application of Laplace's theorem in calculating and proving certain high-order determinants, emphasizing its efficiency in reducing the order of determinants. The key contributions lie in providing a concise method for calculating high-order determinants and proving related propositions using Laplace's theorem, thereby facilitating the learning and application of this powerful tool in linear algebra [9]. This paper studies the spectral radius of uniform hypergraphs determined by the signless Laplacian matrix, deriving bounds and characterizing extremal hypergraphs. The key contributions are novel bounds on the spectral radius and the identification of hypergraphs that attain these bounds, advancing spectral hypergraph theory [10]. This paper analyzes perfect state transfer on edge-corona graphs using the signless Laplacian matrix, uncovering conditions for its absence. The key contributions lie in providing spectral decompositions and identifying sufficient conditions for the non-existence of perfect state transfer [11]. This paper presents sufficient conditions involving spectral spreads and eigenratios for k-connected graphs to possess Hamiltonian s-properties. The key contributions are the derivation of novel spectral conditions that guarantee the existence of Hamiltonian s-properties in k-connected graphs [12].

2. Preliminaries

The numbers that belong to the following set are called hyperbolic complex numbers:

$$\mathbb{D} := \{\delta = x + jy : x, y \in \mathbb{R}\} \tag{1}$$

In this set there are hyperbolic negative units j in which j is satisfied $j^2 = 1$ and $j \neq \pm 1$.

In this set we define two kinds of operations one is an addition operation, the other is a multiplication operation, first we define two hyperbolic complex numbers z_1 and z_2 that satisfy $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ respectively.

The addition operation is as follows:

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + (y_1 + y_2)\, j \end{aligned} \tag{2}$$

The multiplication operation is as follows:

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= (x_1 x_2 + y_1 y_2) + (x_1 y_2 + x_2 y_1)\, j. \end{aligned} \tag{3}$$

For any $z = x + \mathrm{j}y$ we also define $\overline{z}$ as a hyperbolic complex number that satisfies $\overline{z} = x - jy$.

In order to standardize the hyperbolic complex number ring, we defined two units of hyperbolic negative numbers to express any hyperbolic complex number in this ring. These two numbers are:

$$e = \frac{1+j}{2} \text{ and } e^{\dagger} = \frac{1-j}{2} \tag{4}$$

Zero factors e and $e^{\dagger}$ have the following algorithm:

$$ee^{\dagger} = 0, e^2 = e \text{ and } e^{\dagger} = \left(e^{\dagger}\right)^2, \tag{5}$$

$$e+e^{\dagger}=1,\ \ e-e^{\dagger}=j \tag{6}$$

So any hyperbolic complex number can be simply represented by e and $e^{\dagger}$ and the result is as follows:

$$z=(x+y)e+(x-y)e^{\dagger}=:ae+be^{\dagger} \tag{7}$$

Therefore, the addition and multiplication operation for any two hyperbolic complex numbers $z_1=a_1e+b_1e^{\dagger}$ and $z_2=a_2e+b_2e^{\dagger}$ as follow:

$$z_1+z_2=(a_1+a_2)e+(b_1+b_2)e^{\dagger}, \tag{8}$$

$$z_1z_2=(a_1a_2)e+(b_1b_2)e^{\dagger}. \tag{9}$$

The set $\mathbb{D}^{+}$ and $\mathbb{D}^{-}$ can be described as:

$$\mathbb{D}^{+}:=\left\{z=ae+be^{\dagger}:a\geq 0,b\geq 0\right\} \tag{10}$$

$$\mathbb{D}^{-}:=\left\{z=ae+be^{\dagger}:a\leq 0,b\leq 0\right\} \tag{11}$$

Where $\mathbb{D}^{+}$ is called the set of non-negative hyperbolic complex numbers, similarly. $\mathbb{D}^{-}$ is called the set of non-positive hyperbolic complex numbers. The concrete representation of this space in the coordinate axis is shown in Figure 1 as following:

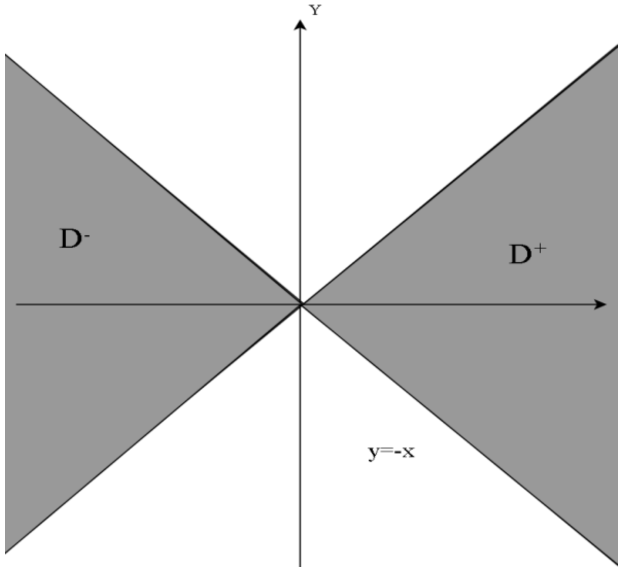

Figure 1. The positive and negative hyperbolic numbers

This article define the relative magnitude of two hyperbolic complex numbers.

For $\forall z_1,z_2\in\mathbb{D}$ we have

$$z_1\preceq z_2\ \text{iff}\ z_2-z_1\in\mathbb{D}^{+}. \tag{12}$$

Similarly

$$z_1\succeq z_2\ \text{iff}\ z_2-z_1\in\mathbb{D}^{-}. \tag{13}$$

Definition 2.1. For $z_1=a_1e+b_1e^{\dagger}$ and $z_2=a_2e+b_2e^{\dagger}$ in $\mathbb{D}$ such that $z_1\preceq z_2$ the closed hyperbolic interval $[z_1,z_2]_{\mathbb{D}}$ is defined by

$$\mathbb{D}_1=[z_1,z_2]_{\mathbb{D}}=\{z_3\in\mathbb{D}:z_1\preceq z_3\preceq z_2\} \tag{14}$$

The range represented by $\mathbb{D}_1$ is shown in Figure 2 below

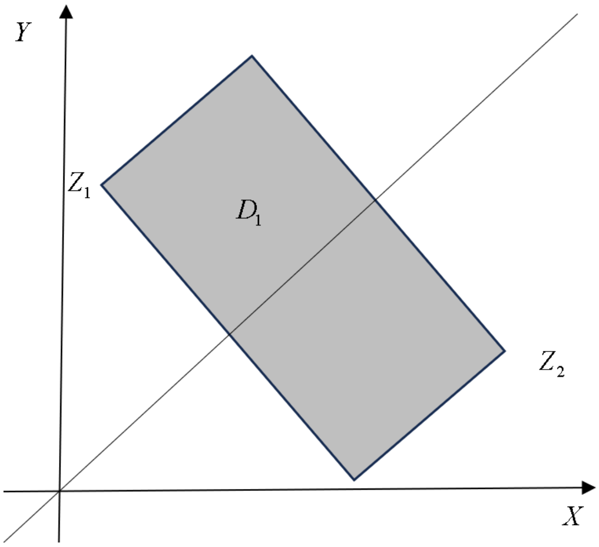

Figure 2. the hyperbolic interval $[z_1, z_2]_{\mathbb{D}}$

3. Results

3.1. Laplace's theorem for hyperbolic complex determinants

Definition 3.1.1 For a square matrix α of order n, the determinant, denoted as $|\alpha|$ the determinant is given by:

$$\begin{aligned}|\alpha| &= \begin{vmatrix} a_{11}e+b_{11}e^{\dagger} & \dots & a_{1n}e+b_{1n}e^{\dagger} \\ \vdots & \ddots & \vdots \\ a_{m1}e+b_{n1}e^{\dagger} & \cdots & a_{nn}e+b_{nn}e^{\dagger} \end{vmatrix} = D(z_1,\cdots,z_n) = (-1)^{\tau(j_1,\cdots,j_n)}\sum z_{j_1 1}\cdots z_{j_n n} \\ &= (-1)^{\tau(j_1,\cdots,j_n)}\sum\left(a_{j_1 1}e+b_{j_1 1}e^{\dagger}\right)\cdots\left(a_{j_n n}e+b_{j_n n}e^{\dagger}\right) \\ &= (-1)^{\tau(j_1,\cdots,j_n)}\sum\left(\prod_{i=1}^{n} a_{j_i i}\right)e + (-1)^{\tau(j_1,\cdots,j_n)}\sum\left(\prod_{i=1}^{n} b_{j_i i}\right)e^{\dagger} \\ &= \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{nn} \end{vmatrix}\mathrm{e} + \begin{vmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{nn} \end{vmatrix}e^{\dagger} \\ &= |D_1|\mathrm{e} + |D_2|e^{\dagger}. \end{aligned} \tag{15}$$

Definition 3.1.2 In an n -th order determinant, by arbitrarily selecting k rows and k columns, a k -th order determinant is formed by the elements located at the intersections of these selected rows and columns, preserving their original arrangement. We choose row $i_1, i_2, \cdots, i_k$ $(i_1 \le i_2 \le \cdots \le i_k)$ and column $j_1, j_2, \cdots, j_k$ $(j_1 \le j_2 \le \cdots \le j_k)$ to obtain the mathematical expression of the K -order subformula as follows:

$$N=\begin{pmatrix} z_{i_1j_1} & \cdots & z_{i_1j_k} \\ \vdots & \ddots & \vdots \\ z_{i_kj_1} & \cdots & z_{i_kj_k} \end{pmatrix}=\begin{pmatrix} a_{i_1j_1}e+b_{i_1j_1}e^{\dagger} & \cdots & a_{i_1j_k}e+b_{i_1j_k}e^{\dagger} \\ \vdots & \ddots & \vdots \\ a_{i_kj_1}e+b_{i_kj_1}e^{\dagger} & \cdots & a_{i_kj_1}e+b_{i_kj_1}e^{\dagger} \end{pmatrix}$$
$$=\begin{pmatrix} a_{i_1j_1} & \cdots & a_{i_1j_k} \\ \vdots & \ddots & \vdots \\ a_{i_kj_1} & \cdots & a_{i_kj_k} \end{pmatrix}e+\begin{pmatrix} b_{i_1j_1} & \cdots & b_{i_1j_k} \\ \vdots & \ddots & \vdots \\ b_{i_kj_1} & \cdots & b_{i_kj_k} \end{pmatrix}e^{\dagger} \tag{16}$$

Definition 3.1.3 The ($n-k$)-th order determinant N' formed by the remaining elements in α after deleting these k rows and k columns in their original order is called the cofactor of the k -th order minor N . The remaining row number is $i_{k+1},i_{k+2},\cdots,i_n \left(i_{k+1}\le i_{k+2}\le\cdots\le i_n\right)$, the column number is $j_{k+1},j_{k+2},\cdots,j_n \left(j_{k+1}\le j_{k+2}\le\cdots\le j_n\right)$, and the cosubformula m of order k is:

$$N'=\begin{pmatrix} z_{i_{k+1}j_{k+1}} & \cdots & z_{i_{k+1}j_n} \\ \vdots & \ddots & \vdots \\ z_{i_nj_{k+1}} & \cdots & z_{i_nj_n} \end{pmatrix}=\begin{pmatrix} a_{i_{k+1}j_{k+1}}e+b_{i_{k+1}j_{k+1}}e^{\dagger} & \cdots & a_{i_{k+1}j_n}e+b_{i_{k+1}j_n}e^{\dagger} \\ \vdots & \ddots & \vdots \\ a_{i_nj_{k+1}}e+b_{i_nj_{k+1}}e^{\dagger} & \cdots & a_{i_nj_n}e+b_{i_nj_n}e^{\dagger} \end{pmatrix}$$
$$=\begin{pmatrix} a_{i_{k+1}j_{k+1}} & \cdots & a_{i_{k+1}j_n} \\ \vdots & \ddots & \vdots \\ a_{i_nj_{k+1}} & \cdots & a_{i_nj_n} \end{pmatrix}e+\begin{pmatrix} b_{i_{k+1}j_{k+1}} & \cdots & b_{i_{k+1}j_n} \\ \vdots & \ddots & \vdots \\ b_{i_nj_{k+1}} & \cdots & b_{i_nj_n} \end{pmatrix}e^{\dagger} \tag{17}$$

Definition 3.1.4 Let the row and column indexes of the K -order subformula N of α in α be $i_1,i_2,\cdots,i_k$. $j_1,j_2,\cdots,j_k$ Then the expression of the subformula B of N called N after the cosubformula N' is preceded by the symbol $(-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)}$ is:

$$B=(-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)}N' \tag{18}$$

Theorem 3.1.5 (Hyperbolic Laplace's Theorem) Set any row $k\,(1\le k\le n-1)$ in the determinant α . The sum of the product of all K -order subformulas N composed of the elements of the k row and their cofactors B is equal to the determinant α , The subformula obtained by taking row k in α is $N_1,N_2,\cdots,N_t$, their cofactors are $B_1,B_2,\cdots,B_t$, The expression of this theorem is:

$$|\alpha|=N_1B_1+N_2B_2+\cdots+N_tB_t \tag{19}$$

Proof:

From Definition 3.1.1, we have:

$$|\alpha|=\begin{vmatrix} a_{11}e+b_{11}e^{\dagger} & \cdots & a_{1n}e+b_{1n}e^{\dagger} \\ \vdots & \ddots & \vdots \\ a_{m1}e+b_{n1}e^{\dagger} & \cdots & a_{nn}e+b_{nn}e^{\dagger} \end{vmatrix}$$
$$=\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{nn} \end{vmatrix}\mathrm{e}+\begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{nn} \end{vmatrix}e^{\dagger} \tag{20}$$

By Laplace's theorem for determinants with real coefficients:

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{nn} \end{vmatrix} = \sum \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k j_1} & \cdots & a_{i_k j_k} \end{vmatrix} (-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)} \begin{vmatrix} a_{i_{k+1} j_{k+1}} & \cdots & a_{i_{k+1} j_n} \\ \vdots & \ddots & \vdots \\ a_{i_n j_{k+1}} & \cdots & a_{i_n j_n} \end{vmatrix}, \tag{21}$$

$$\begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{nn} \end{vmatrix} = \sum \begin{vmatrix} b_{i_1 j_1} & \cdots & b_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ b_{i_k j_1} & \cdots & b_{i_k j_k} \end{vmatrix} (-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)} \begin{vmatrix} b_{i_{k+1} j_{k+1}} & \cdots & b_{i_{k+1} j_n} \\ \vdots & \ddots & \vdots \\ b_{i_n j_{k+1}} & \cdots & b_{i_n j_n} \end{vmatrix}. \tag{22}$$

From the given information(20)、(21)、(22), we can deduce

$$\begin{aligned} |\alpha| = \sum & \begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k j_1} & \cdots & a_{i_k j_k} \end{vmatrix} (-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)} \begin{vmatrix} a_{i_{k+1} j_{k+1}} & \cdots & a_{i_{k+1} j_n} \\ \vdots & \ddots & \vdots \\ a_{i_n j_{k+1}} & \cdots & a_{i_n j_n} \end{vmatrix} \mathrm{e} + \\ \sum & \begin{vmatrix} b_{i_1 j_1} & \cdots & b_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ b_{i_k j_1} & \cdots & b_{i_k j_k} \end{vmatrix} (-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)} \begin{vmatrix} b_{i_{k+1} j_{k+1}} & \cdots & b_{i_{k+1} j_n} \\ \vdots & \ddots & \vdots \\ b_{i_n j_{k+1}} & \cdots & b_{i_n j_n} \end{vmatrix} e^{\dagger} \end{aligned} \tag{23}$$

By applying the property of zero divisors (5)、(6), it can be simplified:

$$\begin{aligned} |\alpha| = \sum & \left(\begin{vmatrix} a_{i_1 j_1} & \cdots & a_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ a_{i_k j_1} & \cdots & a_{i_k j_k} \end{vmatrix} e + \begin{vmatrix} b_{i_1 j_1} & \cdots & b_{i_1 j_k} \\ \vdots & \ddots & \vdots \\ b_{i_k j_1} & \cdots & b_{i_k j_k} \end{vmatrix} e^{\dagger} \right) (-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)} \\ & \left(\begin{vmatrix} a_{i_{k+1} j_{k+1}} & \cdots & a_{i_{k+1} j_n} \\ \vdots & \ddots & \vdots \\ a_{i_n j_{k+1}} & \cdots & a_{i_n j_n} \end{vmatrix} e + \begin{vmatrix} b_{i_{k+1} j_{k+1}} & \cdots & b_{i_{k+1} j_n} \\ \vdots & \ddots & \vdots \\ b_{i_n j_{k+1}} & \cdots & b_{i_n j_n} \end{vmatrix} e^{\dagger} \right), \end{aligned} \tag{24}$$

we can conclude that:

$$|\alpha| = N_1 B_1 + N_2 B_2 + \cdots + N_{\mathrm{t}} B_{\mathrm{t}} \tag{25}$$

4. Conclusions

This paper offer a comprehensive understanding of the Laplace theorem within the context of hyperbolic complex determinants. The study reveals that every hyperbolic complex determinant can be uniquely factorized using zero-divisor factorization, which is a fundamental property that sets the stage for further exploration. By delving into the decomposition properties of minors and algebraic cofactors, the research demonstrates how these elements interact within the framework of hyperbolic complex numbers. Notably, the Laplace theorem for hyperbolic complex matrices is proven based on the decomposition of minors and algebraic cofactors. This theorem asserts that the determinant of a hyperbolic complex matrix can be expressed as the sum of the products of the elements of any row (or column) with their corresponding cofactors. This finding is significant because it provides a systematic approach to calculating determinants of higher-order and partitioned hyperbolic complex matrices, thereby facilitating complex algebraic computations. In summary, this paper contributes to the field of hyperbolic complex analysis by establishing the Laplace theorem for hyperbolic complex determinants. The unique decomposition properties and the applicability of the Laplace theorem to higher-order matrices demonstrate the theoretical depth and practical utility of this research. These findings have the potential to impact various fields, including physics, where hyperbolic complex numbers play a crucial role in modeling and analysis.

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References

[1] Ozturk, I., Ozdemir, M. Affine transformations of hyperbolic number plane [J]. Boletin de la Sociedad Matematica Mexicana,2022, 28 (2): 61-79.

[2] Yang Liu. Riemann-Hilbert boundary value problem for generalized regular functions in multiple hyperbolic complex numbers [J]. Journal of Ningxia University, 2006,: 144-146.

[3] Richter, D. On Hyperbolic Complex Numbers. Applied Sciences [J],2022,12 (12):1-21.

[4] Luna-Elizarraras, M. E, Golberg, A.. More about bicomplex Mobius transformations. Advances in Applied Clifford Algebras [J], 2024,*34* (2), 1-35.

[5] Zhao Chengbing, Liu Danxiu, and Liu Huihui. Several theorems on Laplace transforms [J]. Journal of Nanyang Institute of Technology, 2021, 13 (02): 122-124.

[6] Feng Yihu and Yang Xingxing. Application of Laplace's Theorem in Determinant Calculation [J]. Journal of Xinzhou Teachers University, 2021, 37 (02): 14-17.

[7] Zhao Ling, Wang Shuquan, and Li Fushan. A New Method for Constructing Determinant Theory [J]. Journal of Qufu Normal University, 2024, 50 (03): 123-128.

[8] Zhao Chun'e, Yang Hongyuan, and Hu Debin. A New Method for Finding Matrix Inverses and Determinants [J]. Journal of Advanced Mathematical Research, 2024, 27 (01): 16-17.

[9] Ma Jiaqi. Using Laplace's Theorem to Calculate Determinants [J]. Mathematics Learning and Research, 2019, (19): 4.

[10] He Fangguo. Spectral Radius of Uniform Hypergraphs Determined by Signless Laplacian Matrices (in English) [J]. Journal of Mathematics, 2024:1-12

[11] Wang Huimei and Wang Weizhong. Perfect State Transfer on Edge-Corona Graphs via the Signless Laplacian [J]. Journal of Shanxi University (Natural Science Edition), 2024,1-10.

[12] Chen Hongzhang, Li Jianxi, and Shao Weici. Hamiltonian s-Properties and (Laplacian) Spectral Spread of k-Connected Graphs (in English) [J]. Advances in Mathematics, 2024, 53 (06): 1181-1187.