Research on Group Actions and Related Theorems

Mingyue Sun *
School of Dalian Royal School, Dalian, China
* Corresponding Author Email: jiangao0522@hotmail.com

Abstract. Group action is a fundamental concept in the field of algebra and has been studied for many years. It arises from the need to understand the symmetries and transformations of mathematical objects. The study of group action is closely related to other branches of mathematics, such as geometry, topology, and number theory. This dissertation explores the fundamental concepts of group actions, providing a comprehensive definition of group actions along with various special cases and inherent properties of groups. It delves into the Orbit-Stabilizer Theorem, detailing the concepts of orbit and stabilizer, the equivalence relations on sets, and the transitive nature of actions. Furthermore, the dissertation includes proofs of several related theorems, reinforcing the theoretical underpinnings of group actions. There are also some research meanings. For example, Group action provides a powerful tool for studying the structure and properties of mathematical objects. It gives your deep insights into its symmetries, invariants, and other important characteristics by analyzing the way a group acts on a set or space. In short, Group action provides a powerful tool for studying the structure and properties of mathematical objects.

Keywords: in the field of algebra, comprehensive definition of group actions, Orbit-Stabilizer Theorem.

1. Introduction

The origin of Group action can be traced back to the study of symmetries in mathematics and physics. It originated from the need to understand how certain transformations or operations can act on a set or object while preserving its essential properties. The group concept is an important part in abstract algebra was developed to formalize and study these symmetries and actions. Over time, the study of Group action has evolved and expanded, finding applications in geometry, number theory, quantum mechanics and other domains [1].

This article explores the fundamental concepts of group actions, defining what they are and illustrating them with examples, for instance the symmetry groupacting on its vertex graph. It highlights various special cases of group actions, notably those of the symmetric and dihedral groups, and examines the properties of groups, including closure, associativity, identity, and converse. Central to the discourse is the theorem of orbital stabilizer, which build a relationship of the orbit size and the stabilizer [2]. elucidating key definitions and properties such as equivalence relations and action transitivity. Several significant theorems are presented, including those linking orbit size to stabilizer size and asserting that elements within the same orbit share equal orbits. This article outlines key concepts related to group action and ownership in the context of mathematical groups [3]. It begins with a definition of group action and provides examples, followed by discussions on group shares and special cases like class actions [4]. The section on group ownership defines associated operations and covers properties such as closure, associativity, and identity, including different methods of group actions. The orbit-stabilizer theorem is introduced, including its definition, proof, and practical applications, along with an explanation of equivalence relations and their transitivity within group operations [5]. Lastly, it addresses the stabilizer of an element, defining its properties, proving its subset nature, and providing illustrative examples [6].

2. Group Action

Definition 2.1. G is a set, and A is a non-empty set. (G, A) , where ga has a unique image for all a in A under $g \in G$, indicates the group action of G on A. Moreover, the relation of action must satisfy following properties.

$$G \times A \rightarrow A$$
$$(g, a) \longmapsto g \cdot a$$

1. for all $g_1, g_2 \in G$,and for all $a \in A$

$$g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$$

2.$1_G \cdot a = a \quad \forall a \in A$

If the author chooses an element $g \in G$ and an element which show $(g, a) \in G \times A$, that denote $g \cdot a$. The author wants to consider certain elements of G as "permutation" of A in some sense.

The element in the blob image group of the set A, if the author chooses $a \in A$, and $g \in G$, then these are combined to produce some elements $g \cdot a \in A$. Think of this as "g takes a to $g \cdot a$"and draw it [7]. As shown in Figure 1.

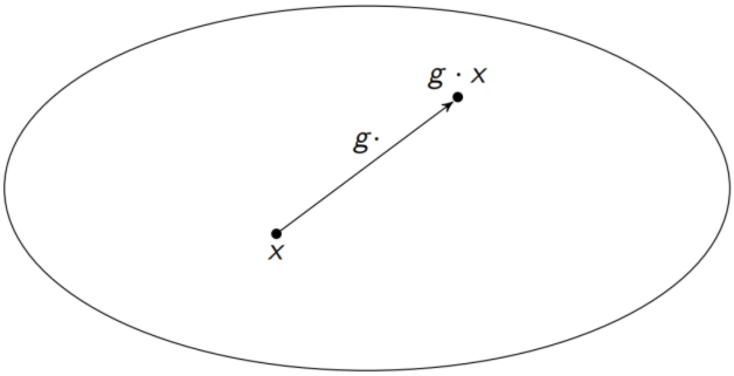

Figure 1. Schematic diagram of the action of group elements on set elements

Example 2.2. Let A be the collection of nodes in a graph, and G is a group which symmetry of A

The author declares G acts on A

$$g \cdot a := g(a).$$

The properties of functions lead to the first axiom;

$$g_1 \cdot (g_2 \cdot x) := g_1(g_2(x)) = (g_1 . g_2)(x) := (g_1 g_2) \cdot x$$

for each and every $g_1, g_2 \in G$ and $a \in A$. Since the identity 1_G is the identity map, the second axiom follows and which fixes all vertices.

Important special cases:

(a) The symmetric group S_n for any set $A = [a_1, a_2, a_3, \dots, a_n]$

S_A acts on A by:

$$\sigma \cdot a = \sigma(a) \ \forall_a \in S_A \text{ and } \forall_a \in A$$

(b) If fixing the numbers of vertex of regular n-gon the group action can be defined by

$$D_{2n} \times [1,2,3, \dots, n] \rightarrow [1,2,3, \dots, n]$$
$$(d, i) \longmapsto d(i)$$

Definition 2.3. Before unclosing the properties of group, we should figure out the concept of group and an associated operation. Then, set G is a group correlation with a binary computation$(G,.) \times G \rightarrow G$ such that establish following axioms:

(i) Closed under group operation:

$$g_1 . g_2 \in G \quad \forall g_1, g_2 \in G$$

(ii) Associativity:

$$(g_1 . g_2) . g_3 = g_1 . (g_2 . g_3). \ \forall g_1, g_2, g_3 \in G$$

(iii) Identity: An identity element $1_G \in G$

$$1_G . g = g \text{ and } g . 1_G = g$$

Every $g \in G$.

(iv) Inverse: Each $g \in G$. there hasan unique inverse element $g^{-1} \in G$

$g^{-1} . g = 1_G$ and $g . g^{-1} = 1_G$

The author will write gh as the simply form for the composition $g . h$.

Definition 2.4. The cardinality of also known as a set G . This indicates that this is the amount of ingredients in G. It is indicated in order of G is denoted $|G|$.

Definition 2.5. There are three popular ways to act on itself by:

(a) 'Left multiplication' is defined $g \cdot h := gh$ for all $g, h \in G$, Thus, the operation is a multiplication to the left by g from the principle is

$$g_1 \cdot (g_2 \cdot h) = g_1 \cdot (g_2 h) = g_1(g_2 h) = (g_1 g_2) \cdot h. = (g_1 g_2)h$$

(b) 'Operation on the right' can be defined $g \cdot h := hg^{-1}$ for all $g \in G, h \in G$. Thus, the action is multiplied g^{-1}on the ringht. It is not that hard to show the operation is a group action.

(c) 'Act by conjugate' can be defined by $g \cdot h := ghg^{-1}$ for all $h, g \in G$

The principle is

$g_1 \cdot (g_2 \cdot h) = g_1 \cdot (g_2 h g_2^{-1}) = g_1 g_2 h g_2^{-1} g_1^{-1} = g_1 g_2 h (g_1 g_2)^{-1} = (g_1 g_2) \cdot h$ for all$g_1, g_2, h \in G$

Moreover, $1_G \cdot h = 1_G h 1_G^{-1} = h$ for all $h \in G$

Every operation of a group G on a group X defines a subgroup, that is, the elements of G Which act trivially on X which is known as kernel of the group action.

Definition 2.6. If $H \subseteq G$, theH is said to be subgroup of G if following axiom hold:

(1) $H \neq \emptyset$

(2) Closed under group operation: $\forall h_1, h_2 \in H$, then $h_1, h_2 \in H$

(3) Unique inverse exits:

$\forall h \in H, \exists!\ h^{-1} \in H$, such that

$$hh^{-1} = h^{-1}h = 1_G$$

Proprosition 2.7. Let G acts on H. Define

$$N := \{g \in G | g \cdot h = h, \text{all } h \in H\}$$

Then N is a subgroup of G.

The author uses a test for subgroup(kernel), in order to confirm this viewpoint. First, $1_G \in N$ since $1_G \cdot h = h$ and $h \in H$, hence $N \neq \emptyset$. If $n_1, n_2 \in N$ then

$$(n_1 n_2) \cdot h = n_1 \cdot (n_2 \cdot h) = n_1 \cdot h = H$$

For all $h \in H$ and so $n_1 n_2 \in N$. Also, if $n \in N$ then

$$h = 1_G \cdot h = (n^{-1} n) \cdot h = n^{-1} \cdot (n \cdot h) = n^{-1} \cdot h$$

For all $h \in H$ and so $n^{-1} \in N$. This show that if $n \in N, n^{-1} \in N$, hence, it is a subgroup. Because N satisfy all axioms from definition2.6.

3. Orbit Stabilizer Theorem

Definition 3.1. G act on A, and let $a \in A$. The stabilizer of a is defined as being

$$\text{StabG}(a) := G_a = \{g \in G \mid g \cdot a = a\}.$$

The author will remove "the G" if it is exact to identify which group the author is referring to.

Definition 3.2. G act on A, and let $a \in A$. G is an orbit group

$$\text{OrbG}(a) = O_a = \{g \cdot a \mid g \in G\}.$$

On the one hand, the orbit is that all the elements of A can be reached by the reader using the elements of g from G by applying, so in the image below, all the symbols of the marked points are found in OrbG (x) As shown in Figure 2:

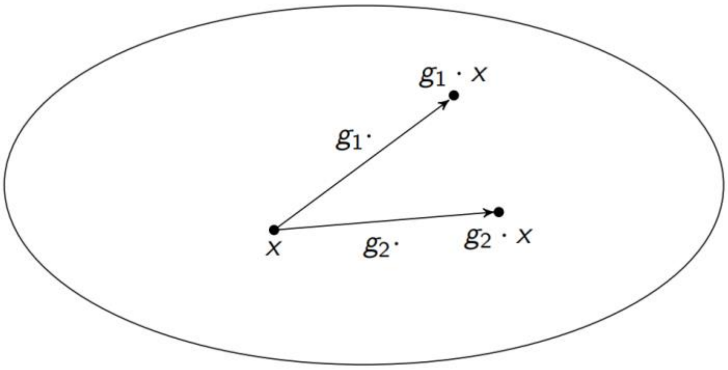

Figure 2. Schematic diagram of the generation of elements in the track

On the other hand, the stabilizer of a G which consist all elements that fixes or does not "move" a, i.e. all those that $g \in G$ for which the image looks like this: as shown in Figure 3.

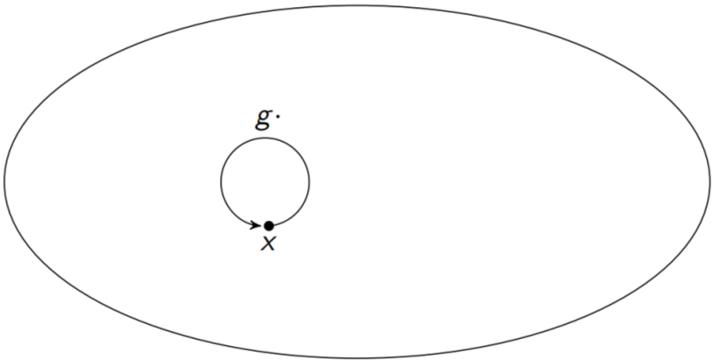

Figure 3. the stabilizer of a G which consist all elements

Example 3.3. Let H be a subgroup and consider the action of multiply on the left. Hence, H acts on G defined by $h \cdot g := hg$. It is easy to show the orbits are precisely the proper classes of H on G.

$g \in G$ is precisely of orbit group

$$\{hg \mid h \in H\} = Hg,$$

Is the right cost of H, moreover, the coset Hg contains g bacause 1_G is in H

Definition 3.4. Let G acts on X, and $x, y \leftarrow X$

$x \sim y$ Only if $y = g \cdot x$ for some $g \in G \Longleftrightarrow y = g \cdot x$

X is equal in relation, and this relation are precisely the different orbits of G corresponding to the group action. Thus, when the author performs makes G acts on X, the author has a separation of the orbits in X.

Definition 3.5. G acts on X transitively by left multiplication, if $\forall x, y \leftarrow X$, $\exists g \in G$ such that $y = g \cdot x$

In another word, there is only one orbit in X under G.

Example 3.6. The D_{2n} group acts on all vertices of X of the $n-$gon transitively.

Let x_1, x_2 be vertices, there must be a rotation element in D_{2n} then of course there has a rotation of d

$$x_1 = d \cdot x_2$$

Theorem 3.7. G is a group of finite order and it acts on a group X, and for all $x \in X$. So, $|\mathrm{OrbG}\ (x)| \times |\mathrm{StabG}\ (x)| = |G|$, or in other words

number of orbits × number of stabilizers = order of G.

As a result, the number of an orbit is a factor of the group G.

Theorem 3.8.

If G acts on the set A by left multiplication, and orbit of an element $a \in A$ contains some other element $b \neq a$, then $\mathrm{orbit}(b) = \mathrm{orbit}(a)$.

Evidence

In order for the element $b \in \text{orbit}(a)$, there must be some element in the group G say g such that $g \cdot a = b$ where $\cdot$ here means the group action of multiplication on the left. Hence, $ga = b$. Then, since the inverse element must exist for g. Thus, $a = g^{-1}b$ and it says that there is some element in G which acts on b to become a. In order to show two sets are equal, first pick an element in orbit(a) say $c = ha$, then $a = h^{-1}c$ and $c = ha = hg^{-1}b$. Since h, g^{-1} are both elements in group G, so hg^{-1} is also in G and this shows that orbit(a) is a subset of orbit(b). On the other hand, pick some d in the orbit b, where $d = h'b$, thus $d = h'b = h'ga$, again $h'g$ is an element in G. This shows another direction of contains holds. Double contains property satisfy and the orbits of a and b are equal.

Definition 3.9. Let G be a group acts on a set A, the stabilizers of an element $a \in A$ is the set of all element's fixes a, formally speaking

$$\text{stabilizer}(a) = \{g \in G, g \cdot a = a\}$$

Theorem 3.10. The stabilizers of any element $a \in A$ forms a subgroup of original group G.

Evidence.

In order to show that $\text{stabilizer}(a)$ is a subgroup, three conditions must be hold,

Firstly, $\text{stabilizer}(a)$ must be hold under group operation. Consider g_1, g_2 are two stabilizers of a which means $g_1 \cdot a = a$ and $g_2 \cdot a = a$, then $(g_1 g_2) \cdot a = g_1 \cdot g_2 \cdot a = g_1 \cdot a = a$. This means $g_1 g_2 \in stabilizer(a)$.

Secondly, the identity elements 1_G must in stabilizer(a), as by the axioms of group actions, $1_G \cdot a = a$ for all a, thus the identity element is in the stabilizers for all elements in A.

Lastly, it must show that the inverse element of a stabilizer must also be a stabilizer of a.

$$g^{-1} \cdot a = g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1_G \cdot a = a$$

Hence, it is clear that the inverse also fixes a if g does.

Theorem 3.11. The Lagrange's Theorem claims that, if H is a subgroup of G, then the order of H must be a factor of the order of group G. Moreover, $\frac{|G|}{|H|}$ equals to quantity of left cosets of the subgroup H.

Theorem 3.11. is very powerful tool to analysis on the structure of the subgroup and constrained on the order of subgroup will eliminante many impossible options by only viewing its order. Unfortunately, the converse for each factor of order G, there exist a subgroup with that order is not ture. However, other powerful theorem such as Cauchy Theorem or Sylow Theorem still provides enough information to identify the structure of subgroup and later on readers can apply such theorems to rule out the extensitence of normal subgroup.

Cauchy Theorem: $|G| = p_1 p_2 \ldots p_n$, where p_i are primes. Then there exists a subgroup of order p_i.

Sylow Theorem: $|G| = ap^k$ where p is not a prime factor of a. There exists a subgroup of order p^k.

4. Conclusion

In short, in this article, we have disscused the concept of group actions and their different properties. First, we defined a group action as an application of the product and a group consist whole, satisfying certain conditions. We also explored the different ways an array can act on itself, such as left, right and conjugate actions. The symmetric group and the dihedral group are presented as important special cases of group actions. We defined the order of a group and showed how each group action defines a subgroup consisting of elements that act trivially as a whole. Furthermore, we introduced the theorem of orbit stabilizer, which relates the orbit size and the size of the stabilizer to the order of the group. We defined the orbit and stabilizer of an element and discussed the equivalence relation in a group induced by a group action. We have also shown that the orbit element is equal to the orbit of every element of the same orbit. Also, we define the stabilizer of an element and demonstrate that it is a subgroup of the set. In general, the study of group actions provides valuable information about the symmetries and transformations of mathematical objects and has applications in various fields of mathematics and science.

References

[1] Jenny Jin. Analysis and Applications of Burnside's Lemma. 2018.
[2] Rahbar Virk. The Orbit-Stabilizer Theorem.
[3] Camilla and David Jordan. Groups. Modular Mathematics. Newnes, 1994.
[4] Micheal Artin. Algebra. 2011.
[5] Jordan, C. R. (Camilla R.), and D. A. (David A.) Jordan. Groups / C.R. Jordan & D.A. Jordan. London: Edward Arnold, 1994. Print.
[6] Algebra, Lecture notes by Clark Barwick, Tudor Dimofte, Ana Rita Pires, Brent Pym, Sue Sierra, and Michael Wemyss. January 6, 2023.
[7] Robinson, D.J.S. (1996) A Course in the Theory of Groups. Springer-Verlag, New York. https://doi.org/10.1007/978 - 1 - 4419 - 8594 - 1.
[8] Wielandt, H. (1964) Finite Permutation Groups. Academic Press, New York.